# CHAPTER 3 CONTINUUM MECHANICS 

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### 3.1 INTRODUCTION

Continuum mechanics is an essential building block of nonlinear finite element analysis, and a mastery of continuum mechanics is essential for a good understanding of nonlinear finite elements. This chapter summarizes the fundamentals of nonlinear continuum mechanics which are needed for a development of nonlinear finite element methods. It is, however, insufficient for thoroughly learning continuum mechanics. Instead, it provides a review of the topics that are particularly relevant to nonlinear finite element analysis. The content of this chapter is limited to topics that are needed for the remainder of the book.

Readers who have little or no familiarity with continuum mechanics should consult texts such as Hodge (1970), Mase and Mase (1992), Fung (1994), Malvern (1969), or Chandrasekharaiah and Debnath (1994). The first three are the most elementary. Hodge (1970) is particularly useful for learning indicial notation and the fundamental topics. Mase and Mase (1992) gives a concise introduction with notation almost identical to that used here. Fung (1994) is an interesting book with many discussions of how continuum mechanics is applied. The text by Malvern (1969) has become a classic in this field for it provides a very lucid and comprehensive description of the field. Chandrasekharaiah and Debnath (1994) gives a thorough introduction with an emphasis on tensor notation. The only topic treated here which is not presented in greater depth in all of these texts is the topic of objective stress rates, which is only covered in Malvern. Monographs of a more advanced character are Marsden and Hughes (1983), Ogden (1984) and Gurtin (). Prager (1961), while an older book, still provides a useful description of continuum mechanics for the reader with an intermediate background. The classic treatise on continuum mechanics is Truesdell and Noll (1965) which discusses the fundamental issues from a very general viewpoint. The work of Eringen (1962) also provides a comprehensive description of the topic.

This Chapter begins with a description of deformation and motion, including some useful equations for characterizing deformation and the time derivatives of variables. Rigid body motion is described with an emphasis on rigid body rotation. Rigid body rotation plays a central role in nonlinear continuum mechanics, and many of the more difficult and complicated aspects of nonlinear continuum mechanics stem from rigid body rotation. The material concerning rigid body rotation should be carefully studied.

Next, the concepts of stress and strain in nonlinear continuum mechanics are described. Stress and strain can be defined in many ways in nonlinear continuum mechanics. We will confine our attention to the strain and stress
measures which are most frequently employed in nonlinear finite element programs. We cover the following kinematic measures in detail: the Green strain tensor and the rate-of-deformation. The second is actually a measure of strain rate, but these two are used in the majority of software. The stress measures treated are: the physical (Cauchy) stress, the nominal stress and the second PiolaKirchhoff stress, which we call PK2 for brevity. There are many others, but frankly even these are too many for most beginning students. The profusion of stress and strain measures is one of the obstacles to understanding nonlinear continuum mechanics. Once one understands the field, one realizes that the large variety of measures adds nothing fundamental, and is perhaps just a manifestation of academic excess. Nonlinear continuum mechanics could be taught with just one measure of stress and strain, but additional ones need to be covered so that the literature and software can be understood.

The conservation equations, which are often called the balance equations, are derived next. These equations are common to both solid and fluid mechanics. They consist of the conservation of mass, momentum and energy. The equilibrium equation is a special case of the momentum equation which applies when there are no accelerations in the body. The conservation equations are derived both in the spatial and the material domains. In a first reading or introductory course, the derivations can be skipped, but the equations should be thoroughly known in at least one form.

The Chapter concludes with further study of the role of rotations in large deformation continuum mechanics. The polar decomposition theorem is derived and explained. Then objective rates, also called frame-invariant rates, of the Cauchy stress tensor are examined. It is shown why rate type constitutive equations in large rotation problems require objective rates and several objective rates frequently used in nonlinear finite elements are presented. Differences between objective rates are examined and some examples of the application of objective rates are illustrated.

### 3.2 DEFORMATION AND MOTION

3.2.1 Definitions. Continuum mechanics is concerned with models of solids and fluids in which the properties and response can be characterized by smooth functions of spatial variables, with at most a limited number of discontinuities. It ignores inhomogeneities such as molecular, grain or crystal structures. Features such as crystal structure sometimes appear in continuum models through the constitutive equations, and an example of this kind of model will be given in Chapter 5, but in all cases the response and properties are assumed to be smooth with a countable number of discontinuities. The objective of continuum mechanics is to provide a description to model the macroscopic behavior of fluids, solids and structures.

Consider a body in an initial state at a time $t=0$ as shown in Fig. 3.1; the domain of the body in the initial state is denoted by $\Omega_{0}$ and called the initial configuration. In describing the motion of the body and deformation, we also need a configuration to which various equations are referred; this is called the reference configuration. Unless we specify otherwise, the initial configuration is used as the reference configuration. However, other configurations can also be used as the reference configuration and we will do so in some derivations. The
significance of the reference configuration lies in the fact that motion is defined with respect to this configuration.


Fig. 3.1. Deformed (current) and undeformed (initial) configurations of a body.
In many cases, we will also need to specify a configuration which is considered to be an undeformed configuration. The notion of an "undeformed" configuration should be viewed as an idealization, since undeformed objects seldom exist in reality. Most objects previously had a different configuration and were changed by deformations: a metal pipe was once a steel ingot, a cellular telephone housing was once a vat of liquid plastic, an airport runway was once a truckload of concrete. So the term undeformed configuration is only relative and designates the configuration with respect to which we measure deformation. In this Chapter, the undeformed configuration is considered to be the initial configuration unless we specifically say otherwise, so it is tacitly assumed that in most cases the initial, reference, and undeformed configurations are identical .

The current configuration of the body is denoted by $\Omega$; this will often also be called the deformed configuration. The domain currently occupied by the body will also be denoted by $\Omega$. The domain can be one, two or three dimensional; $\Omega$ then refers to a line, an area, or a volume, respectively. The boundary of the domain is denoted by $\Gamma$, and corresponds to the two end-points of a segment in one dimension, a curve in two dimensions, and a surface in three dimensions. The developments which follow hold for a model of any dimension from one to three. The dimension of a model is denoted by $n_{S D}$, where "SD" denotes space dimensions.

For a Lagrangian finite element mesh, the initial mesh is a discrete model of the initial, undeformed configuration, which is also the reference configuration. The configurations of the solution meshes are the current, deformed configurations. In an Eulerian mesh, the correspondence is more difficult to picture and is deferred until later.
3.2.2 Eulerian and Lagrangian Coordinates. The position vector of a material point in the reference configuration is given by $\mathbf{X}$, where

$$
\begin{equation*}
\mathbf{X}=X_{i} \mathbf{e}_{i} \equiv \sum_{i=1}^{n_{S D}} X_{i} \mathbf{e}_{i} \tag{3.2.1}
\end{equation*}
$$

where $X_{i}$ are the components of the position vector in the reference configuration and $\mathbf{e}_{\mathrm{i}}$ are the unit base vectors of a rectangular Cartesian coordinate system; indicial notation as described in Section 1.3 has been used in the second expression and will be used throughout this book. Some authors, such as Malvern (1969), also define material particles and carefully distinguish between material points and particles in a continuum. The notion of particles in a continuum is somewhat confusing, for the concept of particles to most of us is discrete rather than continuous. Therefore we will refer only to material points of the continuum.

The vector variable $\mathbf{X}$ for a given material point does not change with time; the variables $\mathbf{X}$ are called material coordinates or Lagrangian coordinates and provide labels for material points. Thus if we want to track the function $f(\mathbf{X}, t)$ at a given material point, we simply track that function at a constant value of $\mathbf{X}$. The position of a point in the current configuration is given by

$$
\begin{equation*}
\mathbf{x}=x_{i} \mathbf{e}_{i} \equiv \sum_{i=1}^{n_{S D}} x_{i} \mathbf{e}_{i} \tag{3.2.2}
\end{equation*}
$$

where $x_{i}$ are the components of the position vector in the current configuration.
3.2.3 Motion. The motion of the body is described by

$$
\begin{equation*}
\mathbf{x}=\phi(\mathbf{X}, t) \quad \text { or } \quad x_{i}=\phi_{i}(\mathbf{X}, t) \tag{3.2.3}
\end{equation*}
$$

where $\mathbf{x}=x_{i} \mathbf{e}_{i}$ is the position at time $t$ of the material point $\mathbf{X}$. The coordinates $\mathbf{x}$ give the spatial position of a particle, and are called spatial, or Eulerian coordinates. The function $\phi(\mathbf{X}, t)$ maps the reference configuration into the current configuration at time $t$., and is often called a mapping or map.

When the reference configuration is identical to the initial configuration, as assumed in this Chapter, the position vector $\mathbf{x}$ of any point at time $t=0$ coincides with the material coordinates, so

$$
\begin{equation*}
\mathbf{X}=\mathbf{x}(\mathbf{X}, 0) \equiv \phi(\mathbf{X}, 0) \quad \text { or } \quad X_{i}=x_{i}(\mathbf{X}, 0)=\phi_{i}(\mathbf{X}, 0) \tag{3.2.4}
\end{equation*}
$$

Thus the mapping $\phi(\mathbf{X}, 0)$ is the identity mapping.
Lines of constant $X_{i}$, when etched into the material, behave just like a Lagrangian mesh; when viewed in the deformed configuration, these lines are no longer Cartesian. Viewed in this way, the material coordinates are often called convected coordinates. In pure shear for example, they become skewed coordinates, just like a Lagrangian mesh becomes skewed, see Fig. 1.2. However, when we view the material coordinates in the reference configuration, they are invariant with time. In the equations to be developed here, the material
coordinates are viewed in the reference configuration, so they are treated as a Cartesian coordinate system. The spatial coordinates, on the other hand, do not change with time regardless of how they are viewed.
3.2.4 Eulerian and Lagrangian Descriptions. Two approaches are used to describe the deformation and response of a continuum. In the first approach, the independent variables are the material coordinates $\mathbf{X}$ and the time $t$, as in Eq. (3.2.3); this description is called a material description or Lagrangian description. In the second approach, the independent variables are the spatial coordinates $\mathbf{x}$ and the time $t$. This is called a spatial or Eulerian description. The duality is similar to that in mesh descriptions, but as we have already seen in finite element formulations, not all aspects of a single formulation are exclusively Eulerian or Lagrangian; instead some finite element formulations combine Eulerian and Lagrangian descriptions as needed.

In fluid mechanics, it is often impossible and unnecessary to describe the motion with respect to a reference configuration. For example, if we consider the flow around an airfoil, a reference configuration is usually not needed for the behavior of the fluid is independent of its history. On the other hand, in solids, the stresses generally depend on the history of deformation and an undeformed configuration must be specified to define the strain. Because of the historydependence of most solids, Lagrangian descriptions are prevalent in solid mechanics.

In the mathematics and continuum mechanics literature, cf. Marsden and Hughes (1983), different symbols are often used for the same field when it is expressed in terms of different independent variables, i.e. when the description is Eulerian or Lagrangian. In this convention, the function which in an Eulerian description is $f(\mathbf{x}, t)$ is denoted by $F(\mathbf{X}, t)$ in a Lagrangian description. The two functions are related by

$$
\begin{equation*}
F(\mathbf{X}, t)=f(\phi(\mathbf{X}, t), t) \text {, or } F=f \circ \phi \tag{3.2.5}
\end{equation*}
$$

This is called a composition of functions; the notation on the right is frequently used in the mathematics literature; see for example $\operatorname{Spivak}(1965$, p.11). The notation for the composition of functions will be used infrequently in this book because it is unfamiliar to most engineers.

The convention of referring to different functions by different symbols is attractive and often adds clarity. However in finite element methods, because of the need to refer to three or more sets of independent variables, this convention becomes quite awkward. Therefore in this book, we associate a symbol with a field, and the specific function is defined by specifying the independent variables. Thus $f(\mathbf{x}, t)$ is the function which describes the field $f$ for the independent variables $\mathbf{x}$ and $t$, whereas $f(\mathbf{X}, t)$ is a different function which describes the same field in terms of the material coordinates. The independent variables are always indicated near the beginning of a section or chapter, and if a change of independent variables is made, the new independent variables are noted.
3.3.5 Displacement, Velocity and Acceleration. The displacement of a material point is given by the difference between its current position and its original position (see Fig. 3.1), so

$$
\begin{equation*}
\mathbf{u}(\mathbf{X}, t)=\phi(\mathbf{X}, t)-\phi(\mathbf{X}, 0)=\phi(\mathbf{X}, t)-\mathbf{X}, \quad u_{i}=\phi_{i}\left(X_{j}, t\right)-X_{i} \tag{3.2.6}
\end{equation*}
$$

where $\mathbf{u}(\mathbf{X}, t)=u_{i} \mathbf{e}_{i}$ and we have used Eq. (3.2.4). The displacement is often written as

$$
\begin{equation*}
\mathbf{u}=\mathbf{x}-\mathbf{X}, \quad u_{i}=x_{i}-X_{i} \tag{3.2.7}
\end{equation*}
$$

where (3.2.1) has been used in (3.2.6) to replace $\phi(\mathbf{X}, t)$ by $\mathbf{x}$. Equation (3.2.7) is somewhat ambiguous since it expresses the displacement as the difference of two variables, $\mathbf{x}$ and $\mathbf{X}$, both of which are generally independent variables. The reader must keep in mind that in expressions such as (3.2.7) the variable $\mathbf{x}$ represents the motion $\mathbf{x}(\mathbf{X}, t) \equiv \phi(\mathbf{X}, t)$.

The velocity $\mathbf{v}(\mathbf{X}, t)$ is the rate of change of the position vector for a material point, i.e. the time derivative with $\mathbf{X}$ held constant. Time derivatives with $\mathbf{X}$ held constant are called material time derivatives; or sometimes material derivatives. Material time derivatives are also called total derivatives. The velocity can be written in the various forms shown below

$$
\begin{equation*}
\mathbf{v}(\mathbf{X}, t)=\dot{\mathbf{u}}=\frac{\partial \phi(\mathbf{X}, t)}{\partial t}=\frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t} \tag{3.2.8}
\end{equation*}
$$

In the above, the variable $\mathbf{x}$ is replaced by the displacement $\mathbf{u}$ in the fourth term by using (3.2.7) and the fact that $\mathbf{X}$ is independent of time. The symbol $D() / D t$ and the superposed dot always denotes a material time derivative in this book, though the latter is often used for ordinary time derivatives when the variable is only a function of time.

The acceleration $\mathbf{a}(\mathbf{X}, t)$ is the rate of change of velocity of a material point, or in other words the material time derivative of the velocity, and can be written in the forms

$$
\begin{equation*}
\mathbf{a}(\mathbf{X}, t)=\frac{D \mathbf{v}}{D t} \equiv \dot{\mathbf{v}}=\frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}=\frac{\partial^{2} \mathbf{u}(\mathbf{X}, t)}{\partial t^{2}} \tag{3.2.9}
\end{equation*}
$$

The above expression is called the material form of the acceleration.
When the velocity is expressed in terms of the spatial coordinates and the time, i.e. in an Eulerian description as in $\mathbf{v}(\mathbf{x}, t)$, the material time derivative is obtained as follows. The spatial coordinates in $\mathbf{v}(\mathbf{x}, t)$ are first expressed as a function of the material coordinates and time by using (3.2.3), giving $\mathbf{v}(\phi(\mathbf{X}, t), t)$. The material time derivative is then obtained by the chain rule:

$$
\begin{equation*}
\frac{D v_{i}}{D t}=\frac{\partial v_{i}(\mathbf{x}, t)}{\partial t}+\frac{\partial v_{i}(\mathbf{x}, t)}{\partial x_{j}} \frac{\partial \phi_{j}(\mathbf{X}, t)}{\partial t}=\frac{\partial v_{i}}{\partial t}+\frac{\partial v_{i}}{\partial x_{j}} v_{j} \tag{3.2.10}
\end{equation*}
$$

where the second equality follows from (3.2.8). The second term on the RHS of (3.2.10) is the convective term, which is also called the transport term. In (3.2.10), the first partial derivative on the RHS is taken with the spatial coordinate fixed. This is called the spatial time derivative. It is tacitly assumed throughout this book that when neither the independent variables nor the fixed variable are explicitly indicated in a partial derivative with respect to time, then the spatial coordinate is fixed and we are referring to the spatial time derivative. On the other hand, when the independent variables are specified as in (3.2.8-9), a partial derivative can specify a material time derivative. Equation (3.2.10) is written in tensor notation as

$$
\begin{equation*}
\frac{D \mathbf{v}}{D t}=\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v} \cdot \nabla \mathbf{v}=\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v} \cdot \operatorname{grad} \mathbf{v} \tag{3.2.11}
\end{equation*}
$$

The material time derivative of any variable which is a function of the spatial variables $\mathbf{x}$ and time $t$ can similarly be obtained by the chain rule. Thus for a scalar function $f(\mathbf{x}, t)$ and a tensor function $\sigma_{i j}(\mathbf{x}, t)$, the material time derivatives are given by

$$
\begin{align*}
& \frac{D f}{D t}=\frac{\partial f}{\partial t}+v_{i} \frac{\partial f}{\partial x_{i}}=\frac{\partial f}{\partial t}+\mathbf{v} \cdot \nabla f=\frac{\partial f}{\partial t}+\mathbf{v} \cdot \operatorname{grad} f  \tag{3.2.12}\\
& \frac{D \sigma_{i j}}{D t}=\frac{\partial \sigma_{i j}}{\partial t}+v_{k} \frac{\partial \sigma_{i j}}{\partial x_{k}}=\frac{\partial \sigma}{\partial t}+\mathbf{v} \cdot \nabla \sigma=\frac{\partial \sigma}{\partial t}+\mathbf{v} \cdot \operatorname{grad} \sigma \tag{3.2.13a}
\end{align*}
$$

where the first term on the RHS of each equation is the spatial time derivative and the second term is the convective term.

It should be remarked that the complete description of the motion is not needed to develop the material time derivative in an Eulerian description. In Eulerian meshes, the motion cannot be defined realistically defined as a function of the material positions in the initial configuration; see Chapter 7. In that case, variables such as the velocity can be developed by describing the motion with respect to a reference configuration that coincides with the configuration at a fixed time $t$.

For this purpose, let the configuration at time fixed time $t=\tau$ be the reference configuration and the position vector at that time, denoted by $\mathbf{X}^{\tau}$, be the reference coordinates. These reference coordinates are given by

$$
\begin{equation*}
\mathbf{X}^{\tau}=\phi(\mathbf{X}, \tau) \tag{3.2.13b}
\end{equation*}
$$

Observe we use an upper case $\mathbf{X}$ since we wish to clearly identify it as an independent variable, and we add the superscript $\tau$ to indicate that these reference coordinates are not the position vectors at the initial time. The motion can be described in terms of these reference coordinates by

$$
\begin{equation*}
\mathbf{x}=\phi^{\tau}\left(\mathbf{X}^{\tau}, t\right) \quad \text { for } t \geq \tau \tag{3.2.13c}
\end{equation*}
$$

Now the arguments used to develop (3.2.10) can be repeated; noting that $\mathbf{v}(\mathbf{x}, t)=\mathbf{v}\left(\phi^{\tau}(\mathbf{X}, t), t\right)$

$$
\begin{equation*}
\frac{D v_{i}}{D t}=\frac{\partial v_{i}(\mathbf{x}, t)}{\partial t}+\frac{\partial v(x, t)}{\partial x_{i}} \frac{\partial \phi_{i}^{\tau}}{\partial t} \tag{3.2.13d}
\end{equation*}
$$

with $t=\tau$. Reference configurations coincident with a configuration other than the initial configuration will also be employed in the development of finite element equations.
3.2.6 Deformation Gradient. The description of deformation and the measure of strain are essential parts of nonlinear continuum mechanics. An important variable in the characterization of deformation is the deformation gradient. The deformation gradient is defined by

$$
\begin{equation*}
F_{i j}=\frac{\partial \phi_{i}}{\partial X_{j}} \equiv \frac{\partial x_{i}}{\partial X_{j}} \quad \text { or } \quad \mathbf{F}=\frac{\partial \phi}{\partial \mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \equiv\left(\nabla_{\mathbf{X}} \phi\right)^{T} \tag{3.2.14}
\end{equation*}
$$

Note in the above that the first index of $F_{i j}$ refers to the component of the deformation, the second to the partial derivative. The order can be remembered by noting that the indices appear in the same order in $F_{i j}$ as in the expression for the partial derivative if it is written horizontally as $\partial \phi_{i} / \partial X_{j}$. The operator $\nabla_{\mathbf{X}}$ is the left gradient with respect to the material coordinates. We will only use the left gradient in this book, but to maintain consistency with the notation of others such as Malvern, we follow his convention exactly. Therefore, the transpose of $\nabla_{\mathbf{X}} \phi$ appears in the above because of the convention on subscripts: for the left gradient, the first subscript is the pertains to the gradient, but in $F_{i j}$ the gradient is associated with the second index. The distinction between left and right gradients is not of importance in this book because we will always use the left gradient, but we adhere to the convention so that our equations are consistent with the continuum mechanics literature. In the terminology of mathematics, the deformation gradient is the Jacobian matrix of the vector function $\phi(\mathbf{X}, t)$.

If we consider an infinitesmal line segment $d \mathbf{X}$ in the reference configuration, then it follows from (3.2.14) that the corresponding line segment $d \mathbf{x}$ in the current configuration is given by

$$
\begin{equation*}
d \mathbf{x}=\mathbf{F} \cdot d \mathbf{X} \text { or } d x_{i}=F_{i j} d X_{j} \tag{3.2.15}
\end{equation*}
$$

In the above expression, the dot could have been omitted between the $\mathbf{F}$ and $d \mathbf{X}$, since the expression is also valid as a matrix expression. We have retained it to conform to our conventionof always explicitly indicating contractions in tensor expressions.

In two dimensions, the deformation gradient in a rectangular coordinate system is given by

$$
\left.\mathbf{F}=\left[\begin{array}{ll}
\frac{\partial x_{1}}{\partial X_{1}} & \frac{\partial x_{1}}{\partial X_{2}}  \tag{3.2.16}\\
\frac{\partial x_{2}}{\partial X_{1}} & \frac{\partial x_{2}}{\partial X_{2}}
\end{array}\right]=\left\lvert\, \begin{array}{ll}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y}
\end{array}\right.\right]
$$

As can be seen in the above, in writing a second-order tensor in matrix form, we use the first index for the row number, the second index for the column number.

The determinant of $\mathbf{F}$ is denoted by $J$ and called the Jacobian determinant or the determinant of the deformation gradient

$$
\begin{equation*}
J=\operatorname{det}(\mathbf{F}) \tag{3.2.17}
\end{equation*}
$$

The Jacobian determinant can be used to relate integrals in the current and reference configurations by

$$
\begin{equation*}
\int_{\Omega} f d \Omega=\int_{\Omega_{0}} f J d \Omega_{0} \text { or in 2D: } \int_{\Omega} f(x, y) d x d y=\int_{\Omega_{0}} f(X, Y) J d X d Y \tag{3.2.18}
\end{equation*}
$$

The material derivative of the Jacobian determinant is given by

$$
\begin{equation*}
\frac{D J}{D t} \equiv \dot{J}=J d i v \mathbf{v} \equiv J \frac{\partial v_{i}}{\partial x_{i}} \tag{3.2.19}
\end{equation*}
$$

The derivation of this formula is left as an exercise.
3.2.6 Conditions on Motion. The mapping $\phi(\mathbf{X}, t)$ which describes the motion and deformation of the body is assumed to satisfy the following conditions:

1. the function $\phi(\mathbf{X}, t)$ is continuous and continuously differentiable except on a finite number of sets of measure zero;
2. the function $\phi(\mathbf{X}, t)$ is one-to-one and onto;
3. the Jacobian determinant satisfies the condition $J>0$.

These conditions ensure that $\phi(\mathbf{X}, t)$ is sufficiently smooth so that compatibility is satisfied, i.e. so there are no gaps or overlaps in the deformed body. The motion and its derivatives can be discontinuous or posses dicontinuous derivatives on sets of measure zero; see Section 1.5, so it is characterized as piecewise continuously differentiable. Sets of measure zero are points in one dimension, lines in two dimensions and planes in three dimensions because a point has zero length, a line has zero area, and a surface has zero volume.

The deformation gradient, i.e. the derivatives of the motion, is generally discontinuous on interfaces between materials. Discontinuities in the motion itself characterize phenomena such as a growing crack. We require the number of discontinuities in a motion and its derivatives to be finite. In fact, in some
nonlinear problems, it has been found that the solutions posses an infinite number of discontinuities, see for example James () and Belytschko, et al (1986). However, these solutions are quite unusual and cannot be treated effectively by finite element methods, so we will not concern ourselves with these types of problems.

The second condition in the above list requires that for each point in the reference configuration $\Omega_{0}$, there is a unique point in $\Omega$ and vice versa. This is a sufficient and necessary condition for the regularity of $\mathbf{F}$, i.e. that $\mathbf{F}$ be invertible. When the deformation gradient $\mathbf{F}$ is regular, the Jacobian determinant $J$ must be nonzero, since the inverse of $\mathbf{F}$ exists if and only if its determinant $J \neq 0$. Thus the second and third conditions are related. We have stated a stronger condition that $J$ be positive rather than just nonzero, which will be seen in Section 3.5.4 to follow from mass conservation.
3.2.7 Rigid Body Rotation and Coordinate Transformations. Rigid body rotation plays a crucial role in the theory of nonlinear continuum mechanics. Many of the complexities which permeate the field arise from rigid body rotation. Furthermore, the decision as to whether linear or nonlinear software is appropriate for a particular linear material problem hinges on the magnitude of rigid body rotations. When the rigid body rotations are large enough to render a linear strain measure invalid, nonlinear software must be used.

A rigid body motion consisting of a translation $\mathbf{x}_{T}(t)$ and a rotation about the origin is written as

$$
\begin{equation*}
\mathbf{x}(\mathbf{X}, t)=\mathbf{R}(t) \cdot \mathbf{X}+\mathbf{x}_{T}(t) \quad x_{i}(\mathbf{X}, t)=R_{i j}(t) X_{j}+x_{T i}(t) \tag{3.2.20}
\end{equation*}
$$

where $\mathbf{R}(t)$ is the rotation tensor, also called a rotation matrix. Because rigid body rotation preserves length, and noting that $d \mathbf{x}_{T}=\mathbf{0}$ in rigid body motion, we have

$$
d \mathbf{x} \cdot d \mathbf{x}=d \mathbf{X} \cdot\left(\mathbf{R}^{T} \cdot \mathbf{R}\right) \cdot d \mathbf{X} \quad d x_{i} d x_{i}=R_{i j} d X_{j} R_{i k} d X_{k}=d X_{j}\left(R_{j i}^{T} R_{i k}\right) d X_{k}
$$

Since the length must stay unchanged in rigid body motion, it follows that

$$
\begin{equation*}
\mathbf{R}^{T} \cdot \mathbf{R}=\mathbf{I} \tag{3.2.20b}
\end{equation*}
$$

and its inverse is given by its transpose:

$$
\begin{equation*}
\mathbf{R}^{-1}=\mathbf{R}^{T} \quad R_{i j}^{-1}=R_{i j}^{T}=R_{j i} \tag{3.2.21}
\end{equation*}
$$

The rotation tensor $\mathbf{R}$ is therefore said to be an orthogonal matrix and any transformation by this matrix, such as $\mathbf{x}=\mathbf{R} \mathbf{X}$, is called an orthogonal transformation. Rotation is an example of an orthogoanl transformation.

A rigid body rotation of a Lagrangian mesh of rectangular elements is shown in Fig. 3.2. As can be seen, in the rigid body rotation, the element edges
are rotated but the angles between the edges remain right angles. The element edges are lines of constant $X$ and $Y$, so when viewed in the deformed configuration, the material coordinates are rotated when the body is rotated as shown in Fig. 3.2.

Specific expressions for the rotation matrix can be obtained in various ways. We obtain it here by relating the components of the vector in $\mathbf{r}$ two different coordinate systems with orthogonal base vectors $\mathbf{e}_{i}$ and $\hat{\mathbf{e}}_{i}$; a two dimensional example is shown in Fig. 3.3. The components in the rotated coordinate system are shown in Fig. 3.3. Since the vector $\mathbf{r}$ is independent of the coordinate system

$$
\begin{equation*}
\mathbf{r}=r_{i} \mathbf{e}_{i}=\hat{r}_{i} \hat{\mathbf{e}}_{i} \tag{3.2.22}
\end{equation*}
$$



Fig. 3.2. A rigid body rotation of a Lagrangian mesh showing the material coordinates when viewed in the reference (initial, undeformed) configuration and the current configuration.


Fig. 3.3. Nomenclature for rotation transformation in two dimensions.
Taking the scalar product of the above with $\mathbf{e}_{j}$ gives

$$
\begin{equation*}
r_{i} \mathbf{e}_{i} \cdot \mathbf{e}_{j}=\hat{r}_{i} \hat{\mathbf{e}}_{i} \cdot \mathbf{e}_{j} \rightarrow r_{i} \delta_{i j}={\hat{r_{i}}}_{i} \hat{\mathbf{e}}_{i} \cdot \mathbf{e}_{j} \rightarrow r_{j}=R_{j i} \hat{r}_{i}, \quad R_{j i}=\mathbf{e}_{j} \cdot \hat{\mathbf{e}}_{i} \tag{3.2.23}
\end{equation*}
$$

The second equation follows from the orthogonality of the base vectors, (3.2.21).
The above shows that the elements of the rotation matrix are given by the scalar products of the corresponding base vectors; thus $R_{12}=\mathbf{e}_{1} \cdot \hat{\mathbf{e}}_{2}$. So the transformation formulas for the components of a vector are

$$
\begin{equation*}
r_{i}=R_{i j} \hat{r}_{j} \equiv R_{i j} \hat{r}_{j}, \quad \hat{r}_{j}=R_{j i}^{T} r_{i}=R_{i j} r_{i} \tag{3.2.24}
\end{equation*}
$$

where the equation on the right follows from (3.3.20b). In the second term of the indicial forms of the equations we have put the hat on the component associated with the hatted coordinates, but later it is often omitted. Note that the hatted index is always the second index of the rotation matrix; this convention helps in remembering the form of the transformation eqaution. In matrix form the above are written as

$$
\mathbf{r}=\mathbf{R} \hat{\mathbf{r}}, \quad \hat{\mathbf{r}}=\mathbf{R}^{T} \mathbf{r}
$$

The above is a matrix expression, as indicated by the absence of dots between the terms. The column matrices of components $\mathbf{r}$ and $\hat{\mathbf{r}}$ differ, but they pertain to the same tensor. In many works, this distinction is clarified by using different symbols for matrices and tensors, but the notation we have chosen does not pemit this distinction.

Writing out the rotation transformation in two dimensions gives

$$
\left\{\begin{array}{l}
r_{x}  \tag{3.2.25}\\
r_{y}
\end{array}\right\}=\left[\begin{array}{ll}
R_{x \hat{x}} & R_{x \hat{y}} \\
R_{y \hat{x}} & R_{y \hat{y}}
\end{array}\right]\left\{\begin{array}{l}
\hat{r}_{x} \\
\hat{r}_{y}
\end{array}\right\}=\left[\begin{array}{ll}
\mathbf{e}_{x} \cdot \mathbf{e}_{\hat{x}} & \mathbf{e}_{x} \cdot \mathbf{e}_{\hat{y}} \\
\mathbf{e}_{y} \cdot \mathbf{e}_{\hat{x}} & \mathbf{e}_{y} \cdot \mathbf{e}_{\hat{y}}
\end{array}\right]\left\{\begin{array}{l}
\hat{r}_{x} \\
\hat{r}_{y}
\end{array}\right\}=\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right]\left\{\begin{array}{l}
\hat{r}_{x} \\
\hat{r}_{y}
\end{array}\right\}
$$

In the above, it can be seen that the subscripts of the rotation matrix correspond to the vector components which are related by that term; for example, in the expression for the $x$ component in row 1 , the $R_{x \hat{y}}$ is the coefficient of the $\hat{y}$ component of $\mathbf{r}$. The last form of the transformation in the above is obtained by evaluating the scalar products from Fig. 3.3 by inspection.

The rotation of a vector is obtained by a similar relation. If the vector $\mathbf{w}$ is obtained by a rotation of the vector $\mathbf{v}$, the two are related by

$$
\begin{equation*}
\mathbf{w}=\mathbf{R} \cdot \mathbf{v}, \quad w_{i}=R_{i j} v_{j} \tag{3.2.26}
\end{equation*}
$$

The first of the above can be written as

$$
\begin{equation*}
\mathbf{w}=\mathbf{R} \cdot\left(v_{j} \mathbf{e}_{j}\right)=v_{j}\left(\mathbf{R} \cdot \mathbf{e}_{j}\right)=v_{j} \hat{\mathbf{e}}_{j} \tag{3.2.27}
\end{equation*}
$$

where we have used the fact that the base vectors transform exactly like the components; this can easily be derived by using (3.2.23). Taking the inner product of the first and last expressions of the above with the rotated base vector $\hat{\mathbf{e}}_{i}$ gives

$$
\begin{equation*}
\hat{w}_{i}=\hat{\mathbf{e}}_{i} \cdot \mathbf{w}=v_{j}\left(\hat{\mathbf{e}}_{i} \cdot \hat{\mathbf{e}}_{j}\right)=v_{j} \boldsymbol{\delta}_{i j}=v_{i} \tag{3.2.28}
\end{equation*}
$$

This shows that the components of the rotated vector $\mathbf{w}$ in the rotated coordinate system are identical to the components of the vector $\mathbf{v}$ in the unrotated coordinate system.

The components of a second order tensor $\mathbf{D}$ are transformed between different coordinate systems by

$$
\begin{equation*}
\mathbf{D}=\mathbf{R} \hat{\mathbf{D}} \mathbf{R}^{T} \quad D_{i j}=R_{i k} \hat{D}_{k l} R_{l j}^{T} \tag{3.2.30a}
\end{equation*}
$$

The inverse of the above is obtained by premultiplying by $\mathbf{R}^{T}$, postmultiplying by $\mathbf{R}$ and using the orthogonality of $\mathbf{R},(3.2 .20 \mathrm{~b})$ :

$$
\begin{equation*}
\hat{\mathbf{D}}=\mathbf{R}^{T} \mathbf{D} \mathbf{R} \quad D_{i j}=R_{i k} \hat{D}_{k l} R_{l j}^{T} \tag{3.2.30b}
\end{equation*}
$$

Note that the above are matrix expressions which relate the components of the same tensor in two different coordinate systems.

The velocity for a rigid body motion can be obtained by taking the time derivative of Eq. (3.2.20). This gives

$$
\begin{equation*}
\dot{\mathbf{x}}(\mathbf{X}, t)=\dot{\mathbf{R}}(t) \cdot \mathbf{X}+\dot{\mathbf{x}}_{T}(t) \quad \text { or } \quad \dot{x}_{i}(\mathbf{X}, t)=\dot{R}_{i j}(t) X_{j}+\dot{x}_{T i}(t) \tag{3.2.31}
\end{equation*}
$$

The structure of rigid body rotation can be clarified by expressing the material coordinates in (3.2.31) in terms of the spatial coordinates via (3.2.20), giving

$$
\begin{equation*}
\mathbf{v} \equiv \dot{\mathbf{x}}=\dot{\mathbf{R}} \cdot \mathbf{R}^{T} \cdot\left(\mathbf{x}-\mathbf{x}_{T}\right)+\dot{\mathbf{x}}_{T} \tag{3.2.32}
\end{equation*}
$$

The tensor

$$
\begin{equation*}
\Omega=\dot{\mathbf{R}} \cdot \mathbf{R}^{T} \tag{3.2.33}
\end{equation*}
$$

is called the angular velocity tensor or angular velocity matrix, Dienes(1979, p 221). It is a skew symmetric tensor, skew symmetric tensors are also called antisymmetric tensors. To demonstrate the skew symmetry of the angular velocity tensor, we take the time derivative of (3.2.21) which gives

$$
\begin{equation*}
\frac{D}{D t}\left(\mathbf{R} \cdot \mathbf{R}^{T}\right)=\frac{D \mathbf{I}}{D t}=\mathbf{0} \rightarrow \dot{\mathbf{R}} \cdot \mathbf{R}^{T}+\mathbf{R} \cdot \dot{\mathbf{R}}^{T}=\mathbf{0} \rightarrow \Omega=-\Omega^{T} \tag{3.2.34}
\end{equation*}
$$

Any skew symmetric tensor can be expressed in terms of the components of a vector, cakked the axial vector, and the corresponding action of that matrix on a vector can be replicated by a cross product, so if $\omega$ if the axial vector of $\Omega$, then

$$
\begin{equation*}
\Omega \mathbf{r}=\omega \times \mathbf{r} \quad \text { or } \quad \Omega_{i j} r_{j}=e_{i j k} \omega_{j} r_{k} \tag{3.2.34b}
\end{equation*}
$$

for any $\mathbf{r}$ and

$$
e_{i j k}=\left\{\begin{array}{c}
1 \text { foran even permutationof } i j k  \tag{3.2.36}\\
-1 \text { for an odd permutationof } i j k \\
0 \text { if anyindex is repeated }
\end{array}\right.
$$

The tensor $e_{i j k}$ is called the alternator tensor or permutation symbol.
The relations between the skew symmetric tensor $\Omega$ and its axial vector $\omega$ are

$$
\begin{equation*}
\omega_{i}=\frac{1}{2} e_{i j k} \Omega_{j k}, \quad \Omega_{i j}=e_{i j k} \omega_{k} \tag{3.2.35}
\end{equation*}
$$

which can be obtained by enforcing (3.2.34b) for all $\mathbf{r}$.
In two dimensions, a skew symmetric tensor has a single independent component and its axial vector is perpendicular to the two dimensional plane of the model, so

$$
\Omega=\left[\begin{array}{cc}
0 & \Omega_{12}  \tag{3.2.37a}\\
-\Omega_{12} & 0
\end{array}\right]=\left[\begin{array}{cc}
0 & -\omega_{3} \\
\omega_{3} & 0
\end{array}\right]
$$

In three dimensions, a skew symmetric tensor has three independent components and which are related to the three components of its axial vector by (3.2.25) giving

$$
\Omega=\left[\begin{array}{ccc}
0 & \Omega_{12} & \Omega_{13}  \tag{3.2.37b}\\
-\Omega_{12} & 0 & \Omega_{23} \\
-\Omega_{13} & -\Omega_{23} & 0
\end{array}\right]=\left[\begin{array}{ccc}
0 & \omega_{3} & -\omega_{2} \\
-\omega_{3} & 0 & \omega_{1} \\
\omega_{2} & -\omega_{1} & 0
\end{array}\right]
$$

When Eq. (3.3.32) is expressed in terms of the angular velocity vector, we have

$$
\begin{align*}
v_{i} \equiv \dot{x}_{i} & =\Omega_{i j}\left(x_{j}-x_{T j}\right)+v_{T i} \\
& =e_{i j k} \omega_{j}\left(x_{k}-x_{T k}\right)+v_{T i} \tag{3.2.38}
\end{align*} \quad \text { or } \quad \mathbf{v} \equiv \dot{\mathbf{x}}=\omega \times\left(\mathbf{x}-\mathbf{x}_{T}\right)+\mathbf{v}_{T}
$$

where we have exchanged $k$ and $j$ in the second line and used $e_{k i j}=e_{i j k}$. The second equation is the well known equation for rigid body motion as given in dynamics texts. The first term on the left hand side is velocity due to the rotation about the point $\mathbf{x}_{T}$ and the second term is the translational velocity of the point $\mathbf{x}_{T}$. Any rigid body velocity can be expressed by (3.2.28).

This concludes the formal discussion of rotation in this Chapter. However, the topic of rotation will reappear in many other parts of this Chapter and this book. Rotation, especially when combined with deformation, is fundamental to nonlinear continuum mechanics, and it should be thoroughly understood by a student of this field.

Corotational Rate-of-Deformation. As we shall see later, in many cases it is convenient to rotate the coordinate at each point of the material with the material. The rate-of-deformation is then expressed in terms of its corotational components $\hat{D}_{i j}$, which can be obtained from the global components by (3.2.30). These components can be obtained directly from the velocity field by

$$
\begin{equation*}
\hat{D}_{i j}=\frac{1}{2}\left(\frac{\partial \hat{v}_{i}}{\partial \hat{x}_{j}}+\frac{\partial \hat{v}_{j}}{\partial \hat{x}_{i}}\right) \equiv \operatorname{sym}\left(\frac{\partial \hat{v}_{i}}{\partial \hat{x}_{j}}\right) \equiv v_{\hat{i}, \hat{j}} \tag{3.2.39}
\end{equation*}
$$

where $\hat{v}_{i} \equiv v_{\hat{i}}$ are the components of the velocity field in the corotational system. the corotational system can be obtained from the polar decomposition theorem to be described later or by other techniques; see section 4.6.

Example 3.1 Rotation and Stretch of Triangular Element. Consider the 3-node triangular finite element shown in Fig. 3.4. Let the motion of the nodes be given by

$$
\begin{array}{ll}
x_{1}(t)=y_{1}(t)=0 \\
x_{2}(t)=2(1+a t) \cos \frac{\pi t}{2}, & y_{2}(t)=2(1+a t) \sin \frac{\pi t}{2}  \tag{E3.1.1}\\
x_{3}(t)=-(1+b t) \sin \frac{\pi t}{2}, & \mathrm{y}_{3}(t)=(1+b t) \cos \frac{\pi t}{2}
\end{array}
$$

Find the deformation function and the Jacobian determinant as a function of time and find the values of $a(t)$ and $b(t)$ such that the Jacobian determinant remains constant.


Fig. 3.4. Motion descrived by Eq. (E3.1.1) with the initial configuration at the left and the deformed configuration at $\mathrm{t}=1$ shown at the right.

In terms of the triangular element coordinates $\xi_{I}$, the configuration of a triangular 3-node, linear displacement element at any time can be written as (see Appendix A if you are not familiar with triangular coordinates)

$$
\begin{align*}
& x(\xi, t)=\sum_{I} x_{I}(t) \xi_{I}=x_{1}(t) \xi_{1}+x_{2}(t) \xi_{2}+x_{3}(t) \xi_{3} \\
& y(\xi, t)=\sum_{I} y_{I}(t) \xi_{I}=y_{1}(t) \xi_{1}+y_{2}(t) \xi_{2}+y_{3}(t) \xi_{3} \tag{E3.1.2}
\end{align*}
$$

In the initial configuration, i.e. at $t=0$ :

$$
\begin{align*}
& X=x(\xi, 0)=X_{1} \xi_{1}+X_{2} \xi_{2}+X_{3} \xi_{3}  \tag{E3.1.3}\\
& Y=y(\xi, 0)=Y_{1} \xi_{1}+Y_{2} \xi_{2}+Y_{3} \xi_{3}
\end{align*}
$$

Substituting the coordinates of the nodes in the undeformed configuration into the above, $X_{1}=X_{3}=0, X_{2}=2, Y_{1}=Y_{2}=0, Y_{3}=1$ yields

$$
\begin{equation*}
X=2 \xi_{2}, \quad Y=\xi_{3} \tag{E3.1.4}
\end{equation*}
$$

In this case, the relations between the triangular coordinates and the material coordinates can be inverted by inspection to give

$$
\begin{equation*}
\xi_{2}=\frac{1}{2} X, \quad \xi_{3}=Y \tag{E3.1.5}
\end{equation*}
$$

Substituting (E3.1.1) and (E3.1.5) into (E3.1.2) gives the following expression for the motion

$$
\begin{align*}
& x(\mathbf{X}, t)=X(1+a t) \cos \frac{\pi t}{2}-Y(1+b t) \sin \frac{\pi t}{2}  \tag{E3.1.6}\\
& y(\mathbf{X}, t)=X(1+a t) \sin \frac{\pi t}{2}+Y(1+b t) \cos \frac{\pi t}{2}
\end{align*}
$$

The deformation gradient is given by Eq.(3.2.16):

$$
\mathbf{F}=\left[\begin{array}{ll}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y}  \tag{E3.1.7}\\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y}
\end{array}\right]=\left[\begin{array}{ll}
(1+a t) \cos \frac{\pi t}{2} & -(1+b t) \sin \frac{\pi t}{2} \\
(1+a t) \sin \frac{\pi t}{2} & (1+b t) \cos \frac{\pi t}{2}
\end{array}\right]
$$

The deformation gradient is a function of time only and at any time constant in the element because the displacement in this element is a linear function of the material coordinates. The determinant of the deformation gradient is given by

$$
\begin{equation*}
J=\operatorname{det}(\mathbf{F})=(1+a t)(1+b t)\left(\cos ^{2} \frac{\pi t}{2}+\sin ^{2} \frac{\pi t}{2}\right) \tag{E3.1.8}
\end{equation*}
$$

When $a=b=0$ the Jacobian determinant remains constant, $J=1$. This is a rotation without deformation. As expected, the Jacobian determinant remains constant since the volume (or area in two dimensions) of anypart of a body does not change in a rigid body motion. The second case in which the Jacobian determinant $J$ remains constant is when $b=-a /(1+a t)$, which corresponds to a deformation in which the area of the element remains constant. This is the type of deformation is called an isochoric deformation; the deformation of incompressible materials is isochoric.

Example 3.2 Consider an element which is rotating at a constant angular velocity $\omega$ about the origin. Obtain the accelerations using both the material and spatial descriptions. Fine the deformation gradient F and its rate.

The motion for a pure rotation about the origin is obtained from Eq. (3.2.20) using the rotation matrix in two dimensions (3.2.25):

$$
\mathbf{x}(t)=\mathbf{R}(t) \mathbf{X} \Rightarrow\left\{\begin{array}{l}
x  \tag{E3.2.1}\\
y
\end{array}\right\}=\left[\begin{array}{cc}
\cos \omega t & -\sin \omega t \\
\sin \omega t & \cos \omega t
\end{array}\right]\left\{\begin{array}{l}
X \\
Y
\end{array}\right\}
$$

where we have used $\theta=\omega t$ to express the motion is a function of time; $\omega$ is the angular velocity of the body. The velocity is obtained by taking the derivative of this motion with respect to time, which gives

$$
\left\{\begin{array}{l}
v_{x}  \tag{E3.2.2}\\
v_{y}
\end{array}\right\}=\left\{\begin{array}{l}
\dot{x} \\
\dot{y}
\end{array}\right\}=\omega\left[\begin{array}{cc}
-\sin \omega t & -\cos \omega t \\
\cos \omega t & -\sin \omega t
\end{array}\right]\left\{\begin{array}{l}
X \\
Y
\end{array}\right\}
$$

The acceleration in the material description is obtained by taking time derivatives of the velocities

$$
\left\{\begin{array}{l}
a_{x}  \tag{E3.2.3}\\
a_{y}
\end{array}\right\}=\left\{\begin{array}{l}
\dot{v}_{x} \\
\dot{v}_{y}
\end{array}\right\}=\omega^{2}\left[\begin{array}{ll}
-\cos \omega t & \sin \omega t \\
-\sin \omega t & -\cos \omega t
\end{array}\right]\left\{\begin{array}{l}
X \\
Y
\end{array}\right\}
$$

To obtain a spatial description for the velocity, the material coordinates $X$ and $Y$ in (E3.2.2) are first expressed in terms of the spatial coordinates $x$ and $y$ by inverting (E3.2.1):

$$
\begin{align*}
\left\{\begin{array}{l}
v_{x} \\
v_{y}
\end{array}\right\} & =\omega\left[\begin{array}{cc}
-\sin \omega t & -\cos \omega t \\
\cos \omega t & -\sin \omega t
\end{array}\right]\left[\begin{array}{cc}
\cos \omega t & \sin \omega t \\
-\sin \omega t & \cos \omega t
\end{array}\right]\left\{\begin{array}{l}
x \\
y
\end{array}\right\} \\
& =\omega\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{l}
x \\
y
\end{array}\right\}=\omega\left\{\begin{array}{c}
-y \\
x
\end{array}\right\} \tag{E3.2.4}
\end{align*}
$$

The material time derivative the velocity field in the spatial description, Eq.(E3.2.4), is obtained via Eq.(3.2.11):

$$
\begin{align*}
\frac{D \mathbf{v}}{D t} & =\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v} \cdot \nabla \mathbf{v}=\left\{\begin{array}{l}
\partial v_{x} / \partial t \\
\partial v_{y} / \partial t
\end{array}\right\}+\left[\begin{array}{ll}
\partial v_{x} / \partial x & \partial v_{x} / \partial y \\
\partial v_{y} / \partial x & \partial v_{y} / \partial y
\end{array}\right]\left\{\begin{array}{l}
v_{x} \\
v_{y}
\end{array}\right\} \\
& =0+\left[\begin{array}{cc}
0 & -\omega \\
\omega & 0
\end{array}\right]\left\{\begin{array}{l}
v_{x} \\
v_{y}
\end{array}\right\}=\omega\left\{\begin{array}{l}
-v_{y} \\
v_{x}
\end{array}\right\} \tag{E3.2.5}
\end{align*}
$$

If we then express the velocity field in (E3.2.5) in terms of the spatial coordinates $x$ and $y$ by Eq.(E3.2.4), we have

$$
\left\{\begin{array}{l}
a_{x}  \tag{E3.2.6}\\
a_{y}
\end{array}\right\}=\omega\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right] \omega\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{l}
x \\
y
\end{array}\right\}=\omega^{2}\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]\left\{\begin{array}{l}
x \\
y
\end{array}\right\}=-\omega^{2}\left\{\begin{array}{l}
x \\
y
\end{array}\right\}
$$

This is the well known result for the centrifugal acceleration: the acceleration vector points toward the center of rotation and its magnitude is $\omega^{2}\left(x^{2}+y^{2}\right)^{\frac{1}{2}}$.

To compare the above with the material form of the acceleration (E3.2.3) we use (E3.2.1) to express the spatial coordinates in (E3.2.6) in terms of the material coordinates, which gives

$$
\left\{\begin{array}{l}
\dot{v}_{x}  \tag{E3.2.7}\\
\dot{v}_{y}
\end{array}\right\}=\omega^{2}\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{cc}
\cos \omega t & -\sin \omega t \\
\sin \omega t & \cos \omega t
\end{array}\right]\left\{\begin{array}{l}
X \\
Y
\end{array}\right\}=\omega^{2}\left[\begin{array}{cc}
-\cos \omega t & \sin \omega t \\
-\sin \omega t & -\cos \omega t
\end{array}\right]\left\{\begin{array}{l}
X \\
Y
\end{array}\right\}
$$

which agrees with Eq. (E3.2.3).

The deformation gradient in obtained from its defintion (3.2.14) and (E3.2.1)

$$
\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{X}}=\mathbf{R}=\left[\begin{array}{cc}
\cos \omega t & -\sin \omega t  \tag{E3.2.8}\\
\sin \omega t & \cos \omega t
\end{array}\right] . \quad \mathbf{F}^{-1}=\left[\begin{array}{cc}
\cos \omega t & \sin \omega t \\
-\sin \omega t & \cos \omega t
\end{array}\right]
$$

Example 3.3 Consider a square 4-node element, with 3 of the nodes fixed as shown in Fig. 3.5. Find the locus of positions of node 3 which results in a vanishing Jacobian.


Figure 3.5. Original configuration of a square element and the locus of points for which $J=0$; a deformed configuration with $J<0$ is also shown.

The displacement field for the rectangular element with all nodes but node 3 fixed is given by the bilinear field

$$
\begin{equation*}
u_{x}(X, Y)=u_{3 x} X Y, \quad u_{y}(X, Y)=u_{3 y} X Y \tag{E3.3.1}
\end{equation*}
$$

Since this element is a square, an isoparametric mapping is not needed. This displacement field vanishes along the two shaded edges. The motion is given by

$$
\begin{align*}
& x=X+u_{x}=X+u_{3 x} X Y \\
& y=Y+u_{y}=Y+u_{3 y} X Y \tag{E3.3.2}
\end{align*}
$$

The deformation gradient is obtained from the above and Eq. (3.2.14):

$$
\mathbf{F}=\left[\begin{array}{cc}
1+u_{3 x} Y & u_{3 x} X  \tag{E3.3.3}\\
u_{3 y} Y & 1+u_{3 y} X
\end{array}\right]
$$

The Jacobian determinant is then

$$
\begin{equation*}
J=\operatorname{det}(\mathbf{F})=1+u_{3 x} Y+u_{3 y} X \tag{E3.3.4}
\end{equation*}
$$

We now examine when the Jacobian determinant will vanish. We need only consider the Jacobian determinant for material particles in the undeformed configuration of the element, i.e. the unit square $X \in[0,1], Y \in[0,1]$. From the Eq. (E3.3.4), it is apparent that $J$ is minimum when $u_{3 x}<0$ and $u_{3 y}<0$. Then the minimum value of $J$ occurs at $X=Y=1$, so

$$
\begin{equation*}
J \geq 0 \Rightarrow 1+u_{3 x} Y+u_{3 y} X \geq 0 \Rightarrow 1+u_{3 x}+u_{3 y} \geq 0 \tag{E3.3.5}
\end{equation*}
$$

The locus of points along which $J=0$ is given by a linear function of the nodal displacements shown in Fig. 3.5, which also shows one deformed configuration of the element for which $J<0$. As can be seen, the Jacobian becomes negative when node 3 crosses the diagonal of the undeformed element.

Example 3.4. The displacement field around a growing crack is given by

$$
\begin{align*}
& u_{x}=k f(r)\left(a+2 \sin ^{2} \frac{\theta}{2}\right) \cos \frac{\theta}{2} \\
& u_{y}=k f(r)\left(b-2 \cos ^{2} \frac{\theta}{2}\right) \sin \frac{\theta}{2}  \tag{E3.4.1}\\
& r^{2}=(X-c t)^{2}+Y^{2}, \quad \theta=\tan ^{-1}(Y / X) \tag{E3.4.2}
\end{align*}
$$

where $a, b, c$, and $k$ are parameters which would be determined by the solution of the governing equations. This displacement field corresponds to a crack opening along the $X$-axis at a velocity $c$; the configuration of the body at two times is shown in Fig. 3.6.


Figure 3.6. The initial uncracked configuration and two subsequent configurations for a crack growing along x -axis.

Find the discontinuity in the displacement along the line $Y=0, X \leq 0$. Does this displacement field conform with the requirements on the motion given in Section 3.2.7?

The motion is $x=X+u_{x}, y=Y+u_{y}$. The discontinuity in the displacement field is found by finding the difference in (E3.4.1) for $\theta=\pi^{-}$and $\theta=\pi^{+}$, which gives

$$
\begin{equation*}
\theta=\pi^{-} \Rightarrow u_{x}=0, u_{y}=k f(r) b \tag{E3.4.3}
\end{equation*}
$$

so the jumps, or discontinuities, in the displacement are

$$
\begin{equation*}
\left\langle u_{x}\right\rangle=0,\left\langle u_{y}\right\rangle=2 k f(r) b \tag{E3.4.4}
\end{equation*}
$$

Everywhere else the displacement field is continuous.
This deformation function meets the criteria given in Section 3.3.6 because the discontinuity occurs along only a line, which is a set of measure zero in a two dimensional problem. From Fig. 3.6 it can be seen that in this deformation, the line behind the crack tip splits into two lines. It is also possible to devise deformations where the line does not separate but a discontinuity occurs in the tangential displacement field. Both types of deformations are now common in nonlinear finite element analysis.

### 3.3 STRAIN MEASURES

In contrast to linear elasticity, many different measures of strain and strain rate are used in nonlinear continuum mechanics. Only two of these measures are considered here:

1. the Green (Green-Lagrange) strain $\mathbf{E}$
2. the rate-of-deformation tensor $\mathbf{D}$, also known as the velocity strain or rate-of-strain.
In the following, these measures are defined and some key properties are given. Many other measures of strain and strain rate appear in the continuum mechanics literature; however, the above are the most widely used in finite element methods. It is sometimes advantageous to use other measures in describing constitutive equations as discussed in Chapter 5, and these other strain measures will be introduced as needed.

A strain measure must vanish in any rigid body motion, and in particular in rigid body rotation. If a strain measure fails to meet this requirement, this strain measure will predict the developnet of nonzero strains, and in turn nonzero stresses, in an initially unstressed body due to rigid body rotation. The key reason why the usual linear strain displacement equations are abandoned in nonlinear theory is that they fail this test. This will be shown in Example 3.6. It will be shown in the following that $\mathbf{E}$ and $\mathbf{D}$ vanish in rigid body motion. A strain measure should satisfy other criteria, i.e. it should increase as the deformation increases, etc. (Hill, ). However, the ability to represent rigid body motion is crucial and indicates when geometrically nonlinear theory must be used.
3.3.1 Green strain tensor. The Green strain tensor $\mathbf{E}$ is defined by

$$
\begin{equation*}
d s^{2}-d S^{2}=2 d \mathbf{X} \cdot \mathbf{E} \cdot d \mathbf{X} \quad \text { or } \quad d x_{i} d x_{i}-d X_{i} d X_{i}=2 d X_{i} E_{i j} d X_{j} \tag{3.3.1}
\end{equation*}
$$

so it gives the change in the square of the length of the material vector $d \mathbf{X}$. Recall the vector $d \mathbf{X}$ pertains to the undeformed configuration. Therefore, the Green strain measures the difference of the square of the length of an infinitesimal segment in the current (deformed) configuration and the reference (undeformed)
configuration. To evaluate the Green strain tensor, we use (3.2.15) to rewrite the LHS of (3.3.1) as

$$
\begin{equation*}
d \mathbf{x} \cdot d \mathbf{x}=(d \mathbf{X} \cdot \mathbf{F}) \cdot(\mathbf{F} \cdot d \mathbf{X})=d \mathbf{X} \cdot\left(\mathbf{F}^{T} \cdot \mathbf{F}\right) \cdot d \mathbf{X} \tag{3.3.2}
\end{equation*}
$$

The above are clearer in indicial notation

$$
d \mathbf{x} \cdot d \mathbf{x}=d x_{i} d x_{i}=F_{i j} d X_{j} F_{i k} d X_{k}=d X_{j} F_{j i}^{T} F_{i k} d X_{k}=d \mathbf{X} \cdot\left(\mathbf{F}^{T} \cdot \mathbf{F}\right) \cdot d \mathbf{X}
$$

Using the above with (3.3.1) and $d \mathbf{X} \cdot d \mathbf{X}=d \mathbf{X} \cdot \mathbf{I} \cdot d \mathbf{X}$ gives

$$
\begin{equation*}
d \mathbf{X} \cdot \mathbf{F}^{T} \cdot \mathbf{F} \cdot d \mathbf{X}-d \mathbf{X} \cdot \mathbf{I} \cdot d \mathbf{X}-d \mathbf{X} \cdot 2 \mathbf{E} \cdot d \mathbf{X}=0 \tag{3.3.3}
\end{equation*}
$$

Factoring out the common terms then yields

$$
\begin{equation*}
d \mathbf{X} \cdot\left(\mathbf{F}^{T} \cdot \mathbf{F}-\mathbf{I}-2 \mathbf{E}\right) \cdot d \mathbf{X}=0 \tag{3.3.4}
\end{equation*}
$$

Since the above must hold for all $d \mathbf{X}$, it follows that

$$
\begin{equation*}
\mathbf{E}=\frac{1}{2}\left(\mathbf{F}^{T} \cdot \mathbf{F}-\mathbf{I}\right) \quad \text { or } \quad E_{i j}=\frac{1}{2}\left(F_{i k}^{T} F_{k j}-\delta_{i j}\right) \tag{3.3.5}
\end{equation*}
$$

The Green strain tensor can also be expressed in terms of displacement gradients by

$$
\begin{equation*}
\mathbf{E}=\frac{1}{2}\left(\left(\nabla_{X} \mathbf{u}\right)^{T}+\nabla_{X} \mathbf{u}+\left(\nabla_{X} \mathbf{u}\right)^{T} \cdot \nabla_{X} \mathbf{u}\right), \quad E_{i j}=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial X_{j}}+\frac{\partial u_{j}}{\partial X_{i}}+\frac{\partial u_{k}}{\partial X_{i}} \frac{\partial u_{k}}{\partial X_{j}}\right) \tag{3.3.6}
\end{equation*}
$$

This expression is derived as follows. We first evaluate $\mathbf{F}^{T} \cdot \mathbf{F}$ in terms of the displacements using indicial notation.

$$
\begin{aligned}
& F_{i k}^{T} F_{k j}= F_{k i} F_{k j}=\frac{\partial x_{k}}{\partial X_{i}} \frac{\partial x_{k}}{\partial X_{j}} \quad(\text { definition of transpose and Eq. (3.2.14)) } \\
&=\left(\frac{\partial u_{k}}{\partial X_{i}}+\frac{\partial X_{k}}{\partial X_{i}}\right)\left(\frac{\partial u_{k}}{\partial X_{j}}+\frac{\partial X_{k}}{\partial X_{j}}\right) \quad(\text { by Eq. (3.2.7)) } \\
&=\left(\frac{\partial u_{k}}{\partial X_{i}}+\delta_{k i}\right)\left(\frac{\partial u_{k}}{\partial X_{j}}+\delta_{k j}\right) \\
&=\left(\frac{\partial u_{i}}{\partial X_{j}}+\frac{\partial u_{j}}{\partial X_{i}}+\frac{\partial u_{k}}{\partial X_{i}} \frac{\partial u_{k}}{\partial X_{j}}+\delta_{i j}\right)
\end{aligned}
$$

Substituting the above into (3.3.5) gives (3.3.6).

To show that the Green strain vanishes in rigid body motion, we consider the deformation function for a general rigid body motion described in Eq. (3.2.20): $\mathbf{x}=\mathbf{R} \cdot \mathbf{X}+\mathbf{x}_{T}$. The deformation gradient $\mathbf{F}$ according to Eq (3.2.14) is then given by $\mathbf{F}=\mathbf{R}$. Using the expression for the Green strain, Eq. (3.3.5). gives

$$
\mathbf{E}=\frac{1}{2}\left(\mathbf{R}^{T} \cdot \mathbf{R}-\mathbf{I}\right)=\frac{1}{2}(\mathbf{I}-\mathbf{I})=\mathbf{0}
$$

where the second equality follows from the orthogonality of the rotation tensor, Eq.(3.2.21). This demonstrates that the Green strain will vanish in any rigid body motion, so it meets an important requirement of a strain measure.
3.3.2 Rate-of-deformation. The second measure of strain to be considered here is the rate-of-deformation $\mathbf{D}$. It is also called the velocity strain and the stretching tensor. In contrast to the Green strain tensor, it is a rate measure of strain.

In order to develop an expression for the rate-of-deformation, we first define the velocity gradient $\mathbf{L}$ by

$$
\begin{align*}
& \mathbf{L}=\frac{\partial \mathbf{v}}{\partial \mathbf{x}}=(\nabla \mathbf{v})^{T}=(\operatorname{grad} \mathbf{v})^{T} \quad \text { or } \quad L_{i j}=\frac{\partial v_{i}}{\partial x_{j}}  \tag{3.3.7}\\
& d \mathbf{v}=\mathbf{L} \cdot d \mathbf{x} \quad \text { or } \quad d v_{i}=L_{i j} d x_{j}
\end{align*}
$$

We have shown several tensor forms of the definition which are frequently seen, but we will primarily use the first or the indicial form. In the above, the symbol $\nabla$ or the abbreviation "grad" preceding the function denotes the spatial gradient of the function, i.e., the derivatives are taken with respect to the spatial coordinates. The symbol $\nabla$ always specifies the spatial gradient unless a different coordinate is appended as a subscript, as in $\nabla_{X}$, which denotes the material gradient.

The velocity gradient tensor can be decomposed into symmetric and skew symmetric parts by

$$
\begin{equation*}
\mathbf{L}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{T}\right)+\frac{1}{2}\left(\mathbf{L}-\mathbf{L}^{T}\right) \text { or } L_{i j}=\frac{1}{2}\left(L_{i j}+L_{j i}\right)+\frac{1}{2}\left(L_{i j}-L_{j i}\right) \tag{3.3.8}
\end{equation*}
$$

This is a standard decomposition of a second order tensor or square matrix: any second order tensor can be expressed as the sum of its symmetric and skew symmetric parts in the above manner; skew symmetry is also known as antisymmetry.

The rate-of-deformation $\mathbf{D}$ is defined as the symmetric part of $\mathbf{L}$, i.e. the first term on the RHS of (3.3.8) and the spin $\mathbf{W}$ is the skew symmetric part of $\mathbf{L}$, i.e. the second term on the RHS of (3.3.8). Using these definitions, we can write

$$
\begin{equation*}
\mathbf{L}=(\nabla \mathbf{v})^{T}=\mathbf{D}+\mathbf{W} \quad \text { or } \quad L_{i j}=v_{i, j}=D_{i j}+W_{i j} \tag{3.3.9}
\end{equation*}
$$

$$
\begin{align*}
& \mathbf{D}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{T}\right) \quad \text { or } \quad D_{i j}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right)  \tag{3.3.10}\\
& \mathbf{W}=\frac{1}{2}\left(\mathbf{L}-\mathbf{L}^{T}\right) \quad \text { or } \quad W_{i j}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}-\frac{\partial v_{j}}{\partial x_{i}}\right) \tag{3.3.11}
\end{align*}
$$

The rate-of-deformation is a measure of the rate of change of the square of the length of infinitesimal material line segments. The definition is

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(d s^{2}\right)=\frac{\partial}{\partial t}(d \mathbf{x} \cdot d \mathbf{x})=2 d \mathbf{x} \cdot \mathbf{D} \cdot d \mathbf{x} \quad \forall d \mathbf{x} \tag{3.3.12}
\end{equation*}
$$

The equivalence of (3.3.10) and (3.3.12) is shown as follows. The expression for the rate-of-deformation is obtained from the above as follows:

$$
\begin{align*}
2 d \mathbf{x} \cdot \mathbf{D} \cdot d \mathbf{x} & =\frac{\partial}{\partial t}(d \mathbf{x}(\mathbf{X}, t) \cdot d \mathbf{x}(\mathbf{X}, t))=2 d \mathbf{x} \cdot d \mathbf{v} \quad(\operatorname{using}(3.2 .8)) \\
& =2 d \mathbf{x} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot d \mathbf{x} \quad \text { by chain rule } \\
& =2 d \mathbf{x} \cdot \mathbf{L} \cdot d \mathbf{x} \quad(\operatorname{using}(3.3 .7)) \\
& =d \mathbf{x} \cdot\left(\mathbf{L}+\mathbf{L}^{T}+\mathbf{L}-\mathbf{L}^{T}\right) \cdot d \mathbf{x} \\
& =d \mathbf{x} \cdot\left(\mathbf{L}+\mathbf{L}^{T}\right) \cdot d \mathbf{x} \tag{3.3.13}
\end{align*}
$$

by antisymmetry of $\mathbf{L}-\mathbf{L}^{T}$; (3.3.10) follows from the last line in (3.3.13) due to the arbitrariness of $d \mathbf{x}$.

In the absence of deformation, the spin tensor and angular velocity tensor are equal, $\mathbf{W}=\Omega$. This is shown as follows. In rigid body motion $\mathbf{D}=\mathbf{0}$, so $\mathbf{L}=\mathbf{W}$ and by integrating Eq. (3.3.7b) we have

$$
\begin{equation*}
\mathbf{v}=\mathbf{W} \cdot\left(\mathbf{x}-\mathbf{x}_{T}\right)+\mathbf{v}_{T} \tag{3.3.14}
\end{equation*}
$$

where $\mathbf{x}_{T}$ and $\mathbf{v}_{T}$ are constants of integration. Comparison with Eq. (3.2.32) then shows that the spin and angular velocity tensors are identical in rigid body rotation. When the body undergoes deformation in addition to rotation, the spin tensor generally differs from the angular velocity tensor. This has important implications on the character of objective stress rates, which are discussed in Section 3.7.
3.3.3. Rate-of-deformation in terms of rate of Green strain. The rate-of-deformation can be related to the rate of the Green strain tensor. To obtain
this relation, we first obtain the material gradient of the velocity field, defined in Eq. (3.3.7b), in terms of the spatial gradient by the chain rule:

$$
\begin{equation*}
\mathbf{L}=\frac{\partial \mathbf{v}}{\partial \mathbf{x}}=\frac{\partial \mathbf{v}}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}}, \quad L_{i j}=\frac{\partial v_{i}}{\partial x_{j}}=\frac{\partial v_{i}}{\partial X_{k}} \frac{\partial X_{k}}{\partial x_{j}} \tag{3.3.15}
\end{equation*}
$$

The definition of the deformation gradient is now recalled, Eq. (3.3.10), $F_{i j}=\partial x_{i} / \partial X_{j}$. Taking the material time derivative of the deformation gradient gives

$$
\begin{equation*}
\dot{\mathbf{F}}=\frac{\partial \mathbf{v}}{\partial \mathbf{X}}, \quad \dot{F}_{i j}=\frac{\partial v_{i}}{\partial X_{j}} \tag{3.3.16}
\end{equation*}
$$

By the chain rule

$$
\begin{equation*}
\frac{\partial x_{i}}{\partial X_{k}} \frac{\partial X_{k}}{\partial x_{j}}=\delta_{i j} \rightarrow F_{i k} \frac{\partial X_{k}}{\partial x_{j}}=\delta_{i j} \rightarrow F_{k j}^{-1}=\frac{\partial X_{k}}{\partial x_{j}}, \quad \mathbf{F}^{-1}=\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \tag{3.3.17}
\end{equation*}
$$

Using the above two equations, (3.3.15) can be rewritten as

$$
\begin{equation*}
\mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}, \quad L_{i j}=\dot{F}_{i k} F_{k j}^{-1} \tag{3.3.18}
\end{equation*}
$$

When the deformation gradient is known, this equation can be used to obtain the rate-of-deformation and the Green strain rate. To obtain a single expression relating these two measures of strain rate, we note that from (3.3.10) and (3.3.18) we have

$$
\begin{equation*}
\mathbf{D}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{T}\right)=\frac{1}{2}\left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}+\mathbf{F}^{-T} \cdot \dot{\mathbf{F}}^{T}\right) \tag{3.3.19}
\end{equation*}
$$

Taking the time derivative of the expression for the Green strain, (3.3.5) gives

$$
\begin{equation*}
\dot{\mathrm{E}}=\frac{1}{2} \frac{D}{D t}\left(\mathbf{F}^{T} \cdot \mathbf{F}-\mathbf{I}\right)=\frac{1}{2}\left(\mathbf{F}^{T} \cdot \dot{\mathbf{F}}+\dot{\mathbf{F}}^{T} \cdot \mathbf{F}\right) \tag{3.3.20}
\end{equation*}
$$

Premultiplying Eq. (3.3.19) by $\mathbf{F}^{T} \mathbf{F}$ and postmultiplying by $\mathbf{F}$ gives

$$
\begin{equation*}
\mathbf{F}^{T} \cdot \mathbf{D} \cdot \mathbf{F}=\frac{1}{2}\left(\mathbf{F}^{T} \cdot \dot{\mathbf{F}}+\dot{\mathbf{F}}^{T} \cdot \mathbf{F}\right) \rightarrow \dot{\mathbf{E}}=\mathbf{F}^{T} \cdot \mathbf{D} \cdot \mathbf{F} \quad \text { or } \quad \dot{E}_{i j}=F_{i k}^{T} D_{k l} F_{l j} \tag{3.3.21}
\end{equation*}
$$

where the last equality follows from Eq. (3.3.20). The above can easily be inverted to yield

$$
\begin{equation*}
\mathbf{D}=\mathbf{F}^{-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1} \quad \text { or } \quad D_{i j}=F_{i k}^{-T} \dot{E}_{k l} F_{l j}^{-1} \tag{3.3.22}
\end{equation*}
$$

As we shall see in Chapter 5, (3.3.22) is an example of a push forward operation, (3.3.21) of the pullback operation. The two measures are two ways of viewing the same tensor: the rate of Green strain is expresses in the reference configuration what the rate-of-deformation expresses in the current configuration. However, the
properties of the two forms are somewhat different. For instance, in Example 3.7 we shall see that the integral of the Green strain rate in time is path independent, whereas the integral of the rate-of-deformation is not path independent.

These formulas could be obtained more easily by starting from the definitions of the Green strain tensor and the rate-of-deformation, Eqs. (3.3.1) and (3.3.9), respectively. However, Eq. (3.3.18), which is very useful, would then be skipped. Therefore the other derivation is left as an exercise, Problem?.

## Example 3.5. Strain Measures in Combined Stretch and

 Rotation. Consider the motion of a body given by$$
\begin{align*}
& x(\mathbf{X}, t)=(1+a t) X \cos \frac{\pi}{2} t-(1+b t) Y \sin \frac{\pi}{2} t  \tag{E3.5.1}\\
& y(\mathbf{X}, t)=(1+a t) X \sin \frac{\pi}{2} t+(1+b t) Y \cos \frac{\pi}{2} t \tag{E3.5.2}
\end{align*}
$$

where $a$ and $b$ are positive constants. Evaluate the deformation gradient $\mathbf{F}$, the Green strain $\mathbf{E}$ and rate-of-deformation tensor as functions of time and examine for $t=0$ and $t=1$.

For convenience, we define

$$
\begin{equation*}
A(t) \equiv(1+a t), B(t) \equiv(1+b t), c \equiv \cos \frac{\pi}{2} t, s \equiv \sin \frac{\pi}{2} t \tag{E3.5.3}
\end{equation*}
$$

The deformation gradient $\mathbf{F}$ is evaluated by Eq.(3.2.10) using (E3.5.1):

$$
\mathbf{F}=\left[\begin{array}{ll}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y}  \tag{E3.5.4}\\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y}
\end{array}\right]=\left[\begin{array}{cc}
A c & -B s \\
A s & B c
\end{array}\right]
$$

The above deformation consists of the simultaneous stretching of the material lines along the $X$ and $Y$ axes and the rotation of the element. The deformation gradient is constant in the element at any time, and the other |measures of strain will also be constant at any time. The Green strain tensor is obtained from (3.3.5), with $\mathbf{F}$ given by (E3.5.4), which gives

$$
\begin{align*}
\mathbf{E} & =\frac{1}{2}\left(\mathbf{F}^{T} \cdot \mathbf{F}-\mathbf{I}\right)=\frac{1}{2}\left(\left[\begin{array}{cc}
A c & A s \\
-B s & B c
\end{array}\right]\left[\begin{array}{cc}
A c & -B s \\
A s & B c
\end{array}\right]-\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]\right) \\
& =\frac{1}{2}\left(\left[\begin{array}{cc}
A^{2} & 0 \\
0 & B^{2}
\end{array}\right\rfloor^{-}\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]\right)=\frac{1}{2}\left\lfloor\begin{array}{cc}
2 a t+a^{2} t^{2} & 0 \\
0 & 2 b t+b^{2} t^{2}
\end{array}\right] \tag{E3.5.5}
\end{align*}
$$

It can be seen that the values of the Green strain tensor correspond to what would be expected from its definition: the line segments which are in the $\mathbf{X}$ and $\mathbf{Y}$ directions are extended by at and $b t$, respectively, so $E_{11}$ and $E_{22}$ are nonzero. The strain $E_{11}=E_{X X}$ is positive when $a$ is positive because the line segment along the $X$ axis is lengthened. The magnitudes of the components of the Green strain
correspond to the engineering measures of strain if the quadratic terms in $a$ and $b$ are negligible. The constants are restricted so that at $>-1$ and $b t>-1$, for otherwise the Jacobian of the deformation becomes negative. When $t=0, \mathbf{x}=\mathbf{X}$ and $\mathbf{E}=\mathbf{0}$.

For the purpose of evaluating the rate-of-deformation, we first obtain the velocity, which is the material time derivative of (E3.5.1):

$$
\begin{align*}
& v_{x}=\left(a c-\frac{\pi}{2} A s\right) X-\left(b s+\frac{\pi}{2} B c\right) Y \\
& v_{y}=\left(a s+\frac{\pi}{2} A c\right) X+\left(b c-\frac{\pi}{2} B s\right) Y \tag{E3.5.6}
\end{align*}
$$

The velocity gradient is given by (3.3.7b),

$$
\mathbf{L}=(\nabla \mathbf{v})^{T}=\left[\begin{array}{ll}
\frac{\partial v_{x}}{\partial x} & \frac{\partial v_{x}}{\partial y}  \tag{E3.5.7}\\
\frac{\partial v_{y}}{\partial x} & \frac{\partial v_{y}}{\partial y}
\end{array}\right]=\left[\begin{array}{cc}
a c-\omega A s & -b s-\omega B c \\
a s+\omega A c & b c-\omega B s
\end{array}\right]
$$

Since at $t=0, x=X, y=Y, c=1, s=0, A=B=1$, so the velocity gradient at $t=0$ is given by

$$
\mathbf{L}=(\nabla \mathbf{v})^{T}=\left[\begin{array}{cc}
a & -\frac{\pi}{2}  \tag{E3.5.8}\\
\frac{\pi}{2} & b
\end{array}\right] \rightarrow \mathbf{D}=\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right], \quad \mathbf{W}=\frac{\pi}{2}\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]
$$

To determine the time history of the rate-of-deformation, we first evaluate the |time derivative of the deformation tensor and the inverse of the deformation |tensor. Recall that $\mathbf{F}$ is given in Eq. (E3.5.4)), from which we obtain

$$
\begin{align*}
& \dot{\mathbf{F}}=\left[\begin{array}{cc}
A_{, t} c-\frac{\pi}{2} A s & -B_{, t} s-\frac{\pi}{2} B c \\
A_{, t} s+\frac{\pi}{2} A c & B_{, t} c-\frac{\pi}{2} B s
\end{array}\right] \quad \mathbf{F}^{-1}=\frac{1}{A B}\left[\begin{array}{cc}
B c & B s \\
-A s & A c
\end{array}\right]  \tag{E3.5.9}\\
& \mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}=\frac{1}{A B}\left[\begin{array}{cc}
B a c^{2}+A b s^{2} & c s(B a-A b) \\
c s(B a-A b) & B a s^{2}+A b c^{2}
\end{array}\right]+\frac{\pi}{2}\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right] \tag{E3.5.10}
\end{align*}
$$

The first term on the RHS is the rate-of-deformation since it is the ymmetric part of the velocity gradient, while the second term is the spin, which is skew symmetric. The rate-of-deformation at $t=1$ is given by

$$
\mathbf{D}=\frac{1}{A B}\left[\begin{array}{cc}
A b & 0  \tag{E3.5.11}\\
0 & B a
\end{array}\right]=\frac{1}{1+a+b+a b}\left[\begin{array}{cc}
b+a b & 0 \\
0 & a+a b
\end{array}\right]
$$

Thus, while in the intermediate stages, the shear velocity-strains are nonzero, in the configuration at $t=1$ only the elongational velocity-strains are nonzero. For comparison, the rate of the Green strain at $t=1$ is given by

$$
\dot{\mathbf{E}}=\left[\begin{array}{cc}
A a & 0  \tag{E3.5.12}\\
0 & B b
\end{array}\right]=\left[\begin{array}{cc}
a+a^{2} & 0 \\
0 & b+b^{2}
\end{array}\right]
$$

Example 3.6 An element is rotated by an angle $\theta$ about the origin. Evaluate the infinitesimal strain (often called the linear strain).

For a pure rotation, the motion is given by (3.2.20), $\mathbf{x}=\mathbf{R} \cdot \mathbf{X}$, where the |translation has been dropped and $\mathbf{R}$ is given in Eq.(3.2.25), so

$$
\left\{\begin{array}{l}
x  \tag{E3.6.1}\\
y
\end{array}\right\}=\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right]\left\{\begin{array}{l}
X \\
Y
\end{array}\right\} \quad\left\{\begin{array}{l}
u_{x} \\
u_{y}
\end{array}\right\}=\left[\begin{array}{cc}
\cos \theta-1 & -\sin \theta \\
\sin \theta & \cos \theta-1
\end{array}\right]\left\{\begin{array}{l}
X \\
Y
\end{array}\right\}
$$

In the definition of the linear strain tensor, the spatial coordinates with respect to |which the derivatives are taken are not specified. We take them with respect to |the material coordinates (the result is the same if we choose the spatial coordinates). The infinitesimal strains are then given by

$$
\begin{equation*}
\varepsilon_{x}=\frac{\partial u_{x}}{\partial X}=\cos \theta-1, \quad \varepsilon_{y}=\frac{\partial u_{y}}{\partial Y}=\cos \theta-1, \quad 2 \varepsilon_{x y}=\frac{\partial u_{x}}{\partial Y}+\frac{\partial u_{y}}{\partial X}=0 \tag{E3.6.2}
\end{equation*}
$$

Thus, if $\theta$ is large, the extensional strains do not vanish. Therefore, the linear |strain tensor cannot be used for large deformation problems, i.e. in geometrically |nonlinear problems.

A question that often arises is how large the rotations can be before a nonlinear analysis is required. The previous example provides some guidance to this choice. The magnitude of the strains predicted in (E3.6.2) are an indication of the error due to the small strain assumption. To get a better handle on this error, we expand $\cos \theta$ in a Taylor's series and substitute into (E3.6.2), which gives

$$
\begin{equation*}
\varepsilon_{x}=\cos \theta-1=1-\frac{\theta^{2}}{2}+O\left(\theta^{4}\right)-1 \approx-\frac{\theta^{2}}{2} \tag{3.3.23}
\end{equation*}
$$

This shows that the error in the linear strain is second order in the rotation. The adequacy of a linear analysis then hinges on how large an error can be tolerated and the magnitudes of the strains of interest. If the strains of interest are of order $10^{-2}$, and $1 \%$ error is acceptable (it almost always is) then the rotations can be of order $10^{-2}$, since the error due to the small strain assumption is of order $10^{-4}$. If the strains of interest are smaller, the acceptable rotations are smaller: for strains of order $10^{-4}$, the rotations should be of order $10^{-3}$ for $1 \%$ error. These guidelines assume that the equilibrium solution is stable, i.e. that buckling is not possible. When buckling is possible, measures which can properly account for large deformations should be used or a stability analysis as described in Chapter 6 should be performed.


Fig. 3.7. An element which is sheared, followed by an extension in the y-direction and then subjected to deformations so that it is returned to its initial configuration.

Example 3.7 An element is deformed through the stages shown in Fig. 3.7. The deformations between these stages are linear functions of time. Evaluate the rate-of-deformation tensor $\mathbf{D}$ in each of these stages and obtain the time integral of the rate-of-deformation for the complete cycle of deformation ending in the undeformed configuration.

Each stage of the deformation is assumed to occur over a unit time interval, so for stage $n, t=n-1$. The time scaling is irrelevant to the results, and we adopt this particular scaling to simplify the algebra. The results would be identical with any other scaling. The deformation function that takes state 1 to | state 2 is

$$
\begin{equation*}
x(\mathbf{X}, t)=X+a t Y, \quad y(\mathbf{X}, t)=Y \quad 0 \leq t \leq 1 \tag{E3.7.1}
\end{equation*}
$$

To determine the rate-of-deformation, we will use Eq. (3.3.18), $\mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$ so we first have to determine $\mathbf{F}, \dot{\mathbf{F}}$ and $\mathbf{F}^{-1}$. These are

$$
\mathbf{F}=\left[\begin{array}{cc}
1 & a t  \tag{E3.7.2}\\
0 & 1
\end{array}\right], \quad \dot{\mathbf{F}}=\left[\begin{array}{ll}
0 & a \\
0 & 0
\end{array}\right], \quad \mathbf{F}^{-1}=\left[\begin{array}{cc}
1 & -a t \\
0 & 1
\end{array}\right]
$$

The velocity gradient and rate of deformation are then given by (3.3.10):

$$
\mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}=\left[\begin{array}{cc}
0 & a \\
0 & 0
\end{array}\right]\left[\begin{array}{cc}
1 & -a t \\
0 & 1
\end{array}\right]=\left[\begin{array}{cc}
0 & a \\
0 & 0
\end{array}\right], \quad \mathbf{D}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{T}\right)=\frac{1}{2}\left[\begin{array}{ll}
0 & a \\
a & 0
\end{array}\right](\mathrm{E} 3.7 .3)
$$

Thus the rate-of-deformation is a pure shear, for both elongational components |vanish. The Green strain is obtained by Eq. (3.3.5), its rate by taking the time derivative

$$
\mathbf{E}=\frac{1}{2}\left(\mathbf{F}^{T} \cdot \mathbf{F}-\mathbf{I}\right)=\frac{1}{2}\left[\begin{array}{cc}
0 & a t  \tag{E3.7.4}\\
a t & a^{2} t^{2}
\end{array}\right], \quad \dot{\mathbf{E}}=\frac{1}{2}\left[\begin{array}{cc}
0 & a \\
a & 2 a^{2} t
\end{array}\right]
$$

The Green strain and its rate include an elongational component, $E_{22}$ which is absent in the rate-of-deformation tensor. This component is small when the constant $a$, and hence the magnitude of the shear, is small.

For the subsequent stages of deformation, we only give the motion, the deformation gradient, its inverse and rate and the rate-of-deformation and Green | strain tensors.
configuration 2 to configuration 3

$$
\begin{align*}
& x(\mathbf{X}, t)=X+a Y, \quad y(\mathbf{X}, t)=(1+b t) Y, \quad 1 \leq \bar{t} \leq 2, \quad t=\bar{t}-1  \tag{E3.7.5a}\\
& \mathbf{F}=\left[\begin{array}{cc}
1 & a \\
0 & 1+b t
\end{array}\right], \dot{\mathbf{F}}=\left[\begin{array}{ll}
0 & 0 \\
0 & b
\end{array}\right], \mathbf{F}^{-1}=\frac{1}{1+b t}\left[\begin{array}{cc}
1+b t & -a \\
0 & 1
\end{array}\right]  \tag{E3.7.5b}\\
& \mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}=\frac{1}{1+b t}\left[\begin{array}{ll}
0 & 0 \\
0 & b
\end{array}\right], \quad \mathbf{D}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{T}\right)=\frac{1}{1+b t}\left[\begin{array}{ll}
0 & 0 \\
0 & b
\end{array}\right]  \tag{E3.7.5c}\\
& \mathbf{E}=\frac{1}{2}\left(\mathbf{F}^{T} \cdot \mathbf{F}-\mathbf{I}\right)=\frac{1}{2}\left\{\begin{array}{ll}
0 & a \\
a & a^{2}+b t(b t+2)
\end{array}\right], \quad \dot{\mathbf{E}}=\frac{1}{2}\left[\begin{array}{ll}
0 & 0 \\
0 & 2 b(b t+1)
\end{array}\right] \tag{E3.7.5d}
\end{align*}
$$

configuration 3 to configuration 4 :

$$
\begin{align*}
& x(\mathbf{X}, t)=X+a(1-t) Y, \quad y(\mathbf{X}, t)=(1+b) Y, \quad 2 \leq \bar{t} \leq 3, \quad t=\bar{t}-2  \tag{E3.7.6a}\\
& \mathbf{F}=\left[\begin{array}{cc}
1 & a(1-t) \\
0 & 1+b
\end{array}\right], \quad \dot{\mathbf{F}}=\left[\begin{array}{cc}
0 & -a \\
0 & 0
\end{array}\right], \quad \mathbf{F}^{-1}=\frac{1}{1+b}\left[\begin{array}{cc}
1+b & a(t-1) \\
0 & 1
\end{array}\right]  \tag{E3.7.6b}\\
& \mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}=\frac{1}{1+b}\left[\begin{array}{cc}
0 & -a \\
0 & 0
\end{array}\right], \quad \mathbf{D}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{T}\right)=\frac{1}{2(1+b)}\left[\begin{array}{cc}
0 & -a \\
-a & 0
\end{array}\right] \tag{E3.7.6c}
\end{align*}
$$

configuration 4 to configuration 5:

$$
\begin{align*}
& x(\mathbf{X}, t)=X, \quad y(\mathbf{X}, t)=(1+b-b t) Y, \quad 3 \leq \bar{t} \leq 4, \quad t=\bar{t}-3  \tag{E3.7.7a}\\
& \mathbf{F}=\left[\begin{array}{cc}
1 & 0 \\
0 & 1+b-b t
\end{array}\right], \quad \dot{\mathbf{F}}=\left[\begin{array}{cc}
0 & 0 \\
0 & -b
\end{array}\right], \mathbf{F}^{-1}=\frac{1}{1+b-b t}\left[\begin{array}{cc}
1+b-b t & 0 \\
0 & 1
\end{array}\right]  \tag{E3.7.7b}\\
& \mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}=\frac{1}{1+b-b t}\left[\begin{array}{cc}
0 & 0 \\
0 & -b
\end{array}\right], \quad \mathbf{D}=\mathbf{L} \tag{E3.7.7c}
\end{align*}
$$

The Green strain in configuration 5 vanishes, since at $\bar{t}=4$ the deformation gradient is the unit tensor, $\mathbf{F}=\mathbf{I}$. The time integral of the rate-of-deformation is given by

$$
\int_{0}^{4} \mathbf{D}(t) d t=\frac{1}{2}\left[\begin{array}{ll}
0 & a  \tag{E3.7.8a}\\
a & 0
\end{array}\right]+\left[\begin{array}{cc}
0 & 0 \\
0 & \ln (1+b)
\end{array}\right]+\frac{1}{2(1+b)}\left[\begin{array}{cc}
0 & -a \\
-a & 0
\end{array}\right]+\left[\begin{array}{cc}
0 & 0 \\
0 & -\ln (1+b)
\end{array}\right]
$$

$$
=\frac{a b}{2(1+b)}\left[\begin{array}{ll}
0 & 1  \tag{E3.7.8b}\\
1 & 0
\end{array}\right]
$$

Thus the integral of the rate-of-deformation over a cycle ending in the initial configuration does not vanish. In other words, while the final configuration in this problem is the undeformed configuration so that a measure of strain should vanish, the integral of the rate-of-deformation is nonzero. This has significant repercussions on the range of applicability of hypoelastic formulations to be described in Sections 5? and 5?. It also means that the integral of the rate-of deformation is not a good measure of total strain. It should be noted the integral over the cycle is close enough to zero for engineering purposes whenever $a$ or $b$ are small. The error in the strain is second order in the deformation, which means it is negligible as long as the strains are of order $10^{-2}$. The integral of the Green strain rate, on the other hand, will vanish in this cycle, since it is the time derivative of the Green strain $\mathbf{E}$, which vanishes in the final undeformed state.

## 3 . 4 STRESS MEASURES

3.4.1 Definitions of Stresses. In nonlinear problems, various stress measures can be defined. We will consider three measures of stress:

1. the Cauchy stress $\sigma$,
2. the nominal stress tensor $\mathbf{P}$;
3. the second Piola-Kirchhoff (PK2) stress tensor $\mathbf{S}$.

The definitions of the first three stress tensors are given in Box 3.1.

## Box 3.1 <br> Definition of Stress Measures



Cauchy stress: $\mathbf{n} \cdot \sigma d \Gamma=d \mathbf{f}=\mathbf{t} d \Gamma$
Nominal stress: $\mathbf{n}_{0} \cdot \mathbf{P} d \Gamma_{0}=d \mathbf{f}=\mathbf{t}_{0} d \Gamma_{0}$
2nd Piola-Kirchhoff stress: $\quad \mathbf{n}_{0} \cdot \mathbf{S} d \Gamma_{0}=\mathbf{F}^{-1} \cdot d \mathbf{f}=\mathbf{F}^{-1} \cdot \mathbf{t}_{0} d \Gamma_{0}$
$d \mathbf{f}=\mathbf{t} d \Gamma=\mathbf{t}_{0} d \Gamma_{0}$

The expression for the traction in terms of the Cauchy stress, Eq. (3.4.1), is called Cauchy's law or sometimes the Cauchy hypothesis. It involves the normal to the current surface and the traction (force/unit area) on the current surface. For this reason, the Cauchy stress is often called the physical stress or true stress. For example, the trace of the Cauchy stress, $\operatorname{trace}(\sigma)=-p \mathbf{I}$, gives the true pressure $p$ commonly used in fluid mechanics. The traces of the stress measures $\mathbf{P}$ and $\mathbf{S}$ do not give the true pressure because they are referred to the undeformed area. We will use the convention that the normal components of the Cauchy stress are positive in tension. The Cauchy stress tensor is symmetric, i.e. $\sigma^{T}=\sigma$, which we shall see follows from the conservation of angular momentum.

The definition of the nominal stress $\mathbf{P}$ is similar to that of the Cauchy stress except that it is expressed in terms of the area and normal of the reference surface, i.e. the underformed surface. It will be shown in Section 3.6.3 that the nominal stress is not symmetric. The transpose of the nominal stress is called the first Piola-Kirchhoff stress. (The nomenclature used by different authors for nominal stress and first Piola-Kirchhoff stress is contradictory; Truesdell and Noll (1965), Ogden (1984), Marsden and Hughes (1983) use the definition given here, Malvern (1969) calls $\mathbf{P}$ the first Piola-Kirchhoff stress.) Since $\mathbf{P}$ is not symmetric, it is important to note that in the definition given in Eq. (3.4.2), the normal is to the left of the tensor $\mathbf{P}$.

The second Piola-Kirchhoff stress is defined by Eq. (3.4.3). It differs from $\mathbf{P}$ in that the force is shifted by $\mathbf{F}^{-1}$. This shift has a definite purpose: it makes the second Piola-Kirchhoff stress symmetric and as we shall see, conjugate to the rate of the Green strain in the sense of power. This stress measure is widely used for path-independent materials such as rubber. We will use the abbreviations PK1 and PK2 stress for the first and second Piola-Kirchhoff stress, respectively.
3.4.2 Transformation Between Stresses. The different stress tensors are interrelated by functions of the deformation. The relations between the stresses are given in Box 3.2. These relations can be obtained by using Eqs. (1-3) along with Nanson's relation (p.169, Malvern(1969)) which relates the current normal to the reference normal by

$$
\begin{equation*}
\mathbf{n} d \Gamma=J \mathbf{n}_{0} \cdot \mathbf{F}^{-1} d \Gamma_{0} \quad n_{i} d \Gamma=J n_{j}^{0} F_{j i}^{-1} d \Gamma_{0} \tag{3.4.5}
\end{equation*}
$$

Note that the nought is placed wherever it is convenient: " 0 " and " $e$ " have invariant meaning in this book and can appear as subscripts or superscripts!

To illustrate how the transformations between different stress measures are obtained, we will develop an expression for the nominal stress in terms of the Cauchy stress. To begin, we equate $d \mathbf{f}$ written in terms of the Cauchy stress and the nominal stress, Eqs. (3.4.2) and (3.4.3), giving

$$
\begin{equation*}
d \mathbf{f}=\mathbf{n} \cdot \sigma d \Gamma=\mathbf{n}_{0} \cdot \mathbf{P} d \Gamma_{0} \tag{3.4.6}
\end{equation*}
$$

Substituting the expression for normal $\mathbf{n}$ given by Nanson's relation, (3.4.5) into (3.4.6) gives

$$
\begin{equation*}
J \mathbf{n}_{0} \cdot \mathbf{F}^{-1} \cdot \sigma d \Gamma_{0}=\mathbf{n}_{0} \cdot \mathbf{P} d \Gamma_{0} \tag{3.4.7}
\end{equation*}
$$

Since the above holds for all $\mathbf{n}_{0}$, it follows that

$$
\begin{align*}
& \mathbf{P}=J \mathbf{F}^{-1} \cdot \sigma \text { or } P_{i j}=J F_{i k}^{-1} \sigma_{k j} \text { or } P_{i j}=J \frac{\partial X_{i}}{\partial x_{k}} \sigma_{k j}  \tag{3.4.8a}\\
& J \sigma=\mathbf{F} \cdot \mathbf{P} \text { or } J \sigma_{i j}=F_{i k} P_{k j} \tag{3.4.8b}
\end{align*}
$$

It can be seen immediately from (3.4.8a) that $\mathbf{P} \neq \mathbf{P}^{T}$, i.e. the nominal stress tensor is not symmetric. The balance of angular momentum, which gives the Cauchy stress tensor to be symmetric, $\sigma=\sigma^{T}$, is expressed as

$$
\begin{equation*}
\mathbf{F} \cdot \mathbf{P}=\mathbf{P}^{T} \cdot \mathbf{F}^{T} \tag{3.4.9}
\end{equation*}
$$

The nominal stress can be related to the PK2 stress by multiplying Eq. (3.4.3) by $\mathbf{F}$ giving

$$
\begin{equation*}
d \mathbf{f}=\mathbf{F} \cdot\left(\mathbf{n}_{0} \cdot \mathbf{S}\right) d \Gamma_{0}=\mathbf{F} \cdot\left(\mathbf{S}^{T} \cdot \mathbf{n}_{0}\right) d \Gamma_{0}=\mathbf{F} \cdot \mathbf{S}^{T} \cdot \mathbf{n}_{0} d \Gamma_{0} \tag{3.4.10}
\end{equation*}
$$

The above is somewhat confusing in tensor notation, so it is rewritten below in indicial notation

$$
\begin{equation*}
d f_{i}=F_{i k}\left(n_{j}^{0} S_{j k}\right) d \Gamma_{0}=F_{i k} S_{k j}^{T} n_{j}^{0} d \Gamma_{0} \tag{3.4.11}
\end{equation*}
$$

The force $d \mathbf{f}$ in the above is now written in terms of the nominal stress using (3.4.2):

$$
\begin{equation*}
d \mathbf{f}=\mathbf{n}_{0} \cdot \mathbf{P} d \Gamma_{0}=\mathbf{P}^{T} \cdot \mathbf{n}_{0} d \Gamma_{0}=\mathbf{F} \cdot \mathbf{S}^{T} \cdot \mathbf{n}_{0} d \Gamma_{0} \tag{3.4.12}
\end{equation*}
$$

where the last equality is Eq. (3.4.10) repeated. Since the above holds for all $\mathbf{n}_{0}$, we have

$$
\begin{equation*}
\mathbf{P}=\mathbf{S} \cdot \mathbf{F}^{T} \quad \text { or } \quad P_{i j}=S_{i k} F_{k j}^{T}=S_{i k} F_{j k} \tag{3.4.13}
\end{equation*}
$$

Taking the inverse transformation of (3.4.8a) and substituting into (3.4.13) gives

$$
\begin{equation*}
\sigma=J^{-1} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T} \quad \text { or } \quad \sigma_{i j}=J^{-1} F_{i k} S_{k l} F_{l j}^{T} \tag{3.4.14a}
\end{equation*}
$$

The above relation can be inverted to express the PK2 stress in terms of the Cauchy stress:

$$
\begin{equation*}
\mathbf{S}=\sqrt{ } \mathbf{F}^{-1} \cdot \sigma \cdot \mathbf{F}^{-T} \quad \text { or } \quad S_{i j}=J F_{i k}^{-1} \sigma_{k l} F_{l j}^{-T} \tag{3.4.14b}
\end{equation*}
$$

The above relations between the PK2 stress and the Cauchy stress, like (3.4.8), depend only on the deformation gradient $\mathbf{F}$ and the Jacobian determinant $J=\operatorname{det}(\mathbf{F})$. Thus, if the deformation is known, the state of stress can always be expressed in terms of either the Cauchy stress $\sigma$, the nominal stress $\mathbf{P}$ or the PK2 stress $\mathbf{S}$. It can be seen from (3.4.14b) that if the Cauchy stress is symmetric, then $\mathbf{S}$ is also symmetric: $\mathbf{S}=\mathbf{S}^{T}$. The inverse relationships to (3.4.8) and (3.4.14) are easily obtained by matrix manipulations.
3.4.3. Corotational Stress and Rate-of-Deformation. In some elements, particularly structural elements such as beams and shells, it is convenient to use the Cauchy stress and rate-of-deformation in corotational form, in which all components are expressed in a coordinate system that rotates with the material. The corotational Cauchy stress, denoted by $\hat{\sigma}$, is also called the rotatedstress tensor (Dill p. 245). We will defer the details of how the rotation and the rotation matrix $\mathbf{R}$ is obtained until we consider specific elements in Chapters 4 and 9. For the present, we assume that we can somehow find a coordinate system that rotates with the material.

The corotational components of the Cauchy stress and the corotational rate-of-deformation are obtained by the standard transformation rule for second order tensors, Eq.(3.2.30):

$$
\begin{equation*}
\hat{\boldsymbol{\sigma}}=\mathbf{R}^{T} \cdot \sigma \cdot \mathbf{R} \quad \text { or } \quad \hat{\sigma}_{i j}=R_{i k}^{T} \sigma_{k l} R_{l j} \tag{3.4.15a}
\end{equation*}
$$

$$
\begin{equation*}
\hat{\mathbf{D}}=\mathbf{R}^{T} \cdot \mathbf{D} \cdot \mathbf{R} \quad \text { or } \quad \hat{D}_{i j}=R_{i k}^{T} D_{k l} R_{l j} \tag{3.4.15b}
\end{equation*}
$$

The corotational Cauchy stress tensor is the same tensor as the Cauchy stress, but it is expressed in terms of components in a coordinate system that rotates with the material. Strictly speaking, from a theoretical viewpoint, a tensor is independent of the coordinate system in which its components are expressed. However, such a fundamentasl view can get quite confusing in an introductory text, so we will superpose hats on the tensor whenever we are referring to its corotational components. The corotational rate-of-deformation is similarly related to the rate-of-deformation.

By expressing these tensors in a coordinate system that rotates with the material, it is easier to deal with structural elements and anisotropic materials. The corotational stress is sometimes called the unrotated stress, which seems like a contradictory name: the difference arises as to whether you consider the hatted coordinate system to be moving with the material (or element) or whether you consider it to be a fixed independent entity. Both viewpoints are valid and the choice is just a matter of preference. We prefer the corotational viewpoint because it is easier to picture, see Example 4.?.

## Box 3.2

Transformations of Stresses

|  | Cauchy Stress <br> $\sigma$ | Nominal Stress <br> $\mathbf{P}$ | 2nd Piola- <br> Kirchhoff <br> Stress $\mathbf{S}$ | Corotational <br> Cauchy <br> Stress $\hat{\sigma}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\sigma$ |  | $J^{-1} \mathbf{F} \cdot \mathbf{P}$ | $J^{-1} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T}$ | $\mathbf{R} \cdot \hat{\sigma} \cdot \mathbf{R}^{T}$ |
| $\mathbf{P}$ | $J \mathbf{F}^{-1} \cdot \sigma$ |  | $\mathbf{S} \cdot \mathbf{F}^{T}$ | $J \mathbf{U}^{-1} \cdot \hat{\sigma} \cdot \mathbf{R}^{T}$ |
| $\mathbf{S}$ | $J \mathbf{F}^{-1} \cdot \sigma \cdot \mathbf{F}^{-T}$ | $\mathbf{P} \cdot \mathbf{F}^{-T}$ |  | $J \mathbf{U}^{-1} \cdot \hat{\sigma} \cdot \mathbf{U}^{-1}$ |
| $\hat{\sigma}$ | $\mathbf{R}^{T} \cdot \sigma \cdot \mathbf{R}$ | $J^{-1} \mathbf{U} \cdot \mathbf{P} \cdot \mathbf{R}$ | $J^{-1} \mathbf{U} \cdot \mathbf{S} \cdot \mathbf{U}$ |  |

Note: $d \mathbf{x}=\mathbf{F} \cdot d \mathbf{X}=\mathbf{R} \cdot \mathbf{U} \cdot d \mathbf{X}$ in deformation,
$\mathbf{U}$ is the strectch tensor, see Sec.5?
$d \mathbf{x}=\mathbf{R} \cdot d \mathbf{X}=\mathbf{R} \cdot d \hat{\mathbf{x}}$ in rotation
Example 3.8 Consider the deformation given in Example 3.2, Eq. (E3.2.1). Let the Cauchy stress in the initial state be given by

$$
\sigma(t=0)=\left[\begin{array}{cc}
\sigma_{x}^{0} & 0  \tag{E3.8.1}\\
0 & \sigma_{y}^{0}
\end{array}\right]
$$

Consider the stress to be frozen into the material, so as the body rotates, the initial stress rotates also, as shown in Fig. 3.8.


Figure 3.8. Prestressed body rotated by $90^{\circ}$.
This corresponds to the behavior of an initial state of stress in a rotating solid, which will be explored further in Section 3.6 Evaluate the PK2 stress, the nominal stress and the corotational stress in the initial configuration and the configuration at $t=\pi / 2 \omega$.

In the initial state, $\mathbf{F}=\mathbf{I}$, so

$$
\mathbf{S}=\mathbf{P}=\hat{\boldsymbol{\sigma}}=\boldsymbol{\sigma}=\left[\begin{array}{cc}
\boldsymbol{\sigma}_{x}^{0} & 0  \tag{E3.8.2}\\
0 & \boldsymbol{\sigma}_{y}^{0}
\end{array}\right]
$$

In the deformed configuration at $t=\frac{\pi}{2 \omega}$, the deformation gradient is given by

$$
\mathbf{F}=\left[\begin{array}{cc}
\cos \pi / 2 & -\sin \pi / 2  \tag{E3.8.3}\\
\sin \pi / 2 & \cos \pi / 2
\end{array}\right]=\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right], \quad J=\operatorname{det}(\mathbf{F})=1
$$

Since the stress is considered frozen in the material, the stress state in the rotated configuration is given by

$$
\sigma=\left[\begin{array}{cc}
\sigma_{y}^{0} & 0  \tag{E3.8.4}\\
0 & \sigma_{x}^{0}
\end{array}\right]
$$

The nominal stress in the configuration is given by Box 3.2:

$$
\mathbf{P}=J \mathbf{F}^{-1} \boldsymbol{\sigma}=\left[\begin{array}{cc}
0 & 1  \tag{E3.8.5}\\
-1 & 0
\end{array}\right]\left[\begin{array}{cc}
\sigma_{y}^{0} & 0 \\
0 & \sigma_{x}^{0}
\end{array}\right]=\left[\begin{array}{cc}
0 & \sigma_{x}^{0} \\
-\sigma_{y}^{0} & 0
\end{array}\right]
$$

Note that the nominal stress is not symmetric. The 2nd Piola-Kirchhoff stress can be expressed in terms of the nominal stress $\mathbf{P}$ by Box 3.2 as follows:

$$
\mathbf{S}=\mathbf{P} \cdot \mathbf{F}^{-T}=\left[\begin{array}{cc}
0 & \sigma_{x}^{0}  \tag{E3.8.6}\\
-\sigma_{y}^{0} & 0
\end{array}\right]\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]=\left[\begin{array}{cc}
\sigma_{x}^{0} & 0 \\
0 & \sigma_{y}^{0}
\end{array}\right]
$$

Since the mapping in this case is a pure rotation, $\mathbf{R}=\mathbf{F}$, so when $t=\frac{\pi}{2 \omega}, \hat{\sigma}=\mathbf{S}$.
This example used the notion that an initial state of stress can be considered in a solid is frozen into the material and rotates with the solid. It showed that in a pure rotation, the PK2 stress is unchanged; thus the PK2 stress behaves as if it were frozen into the material. This can also be explained by noting that the material coordinates rotate with the material and the components of the PK2 stress are related to the orientation of the material coordiantes. Thus in the previous example, the component $S_{11}$, which is associated with $X$ components, corresponds to the $\sigma_{22}$ components of physical stress in the final configuration and the components $\sigma_{11}$ in the initial configuration. The corotational components of the Cauchy stress $\hat{\sigma}$ are also unchanged by the rotation of the material, and in the absence of deformation equal the components of the PK2 stress. If the motion were not a pure rotation, the corotational Cauchy stress components would differ from the components of the PK2 stress in the final configuration.

The nominal stress at $t=1$ is more difficult to interpret physically. This stress is kind of an expatriate, living partially in the current configuration and partially in the reference configuration. For this reason, it is often described as a two-point tensor, with a leg in each configuration, the reference configuration and the current configuration. The left leg is associated with the normal in the reference configuration, the right leg with a force on a surface element in the current configuration, as seen from in its defintion, Eq. (3.4.2). For this reason and the lack of symmetry of the nominal stress $\mathbf{P}$, it is seldom used in constitutive equations. Its attractiveness lies in the simplicity of the momentum and finite element equations when expressed in terms of $\mathbf{P}$.

## Example 3.9 Uniaxial Stress.



Figure 3.9. Undeformed and current configurations of a body in a uniaxial state of stress.

Consider a bar in a state of uniaxial stress as shown in Fig. 3.9. Relate the nominal stress and the PK2 stress to the uniaxial Cauchy stress. The initial dimensions (the dimensions of the bar in the reference configuration) are $l_{0}, a_{0}$ and $b_{0}$, and the current dimensions are $l, a$ so

$$
\begin{equation*}
x=\frac{\ell}{\ell_{0}} X, \quad y=\frac{a}{a_{0}} Y, \quad z=\frac{b}{b_{0}} Z \tag{E3.9.1}
\end{equation*}
$$

Therefore

$$
\left.\left.\left.\begin{array}{l}
\left.\mathbf{F}=\begin{array}{lll}
\lceil\partial x / \partial X & \partial x / \partial Y & \partial x / \partial Z\rceil \\
\mid \partial y / \partial X & \partial y / \partial Y & \partial y / \partial Z
\end{array}|=| \begin{array}{ccc}
\ell / \ell_{0} & 0 & 0 \\
0 & a / a_{0} & 0 \\
\partial z / \partial X & \partial z / \partial Y & \partial z / \partial Z
\end{array}\right] \\
0 \\
0
\end{array}\right] b / b_{0}\right\rfloor\right] ~\left[\begin{array}{ll} 
\\
J & =\operatorname{det}(\mathbf{F})=\frac{a b \ell}{a_{0} b_{0} \ell_{0}}  \tag{E3.9.4}\\
\mathbf{F}^{-1}=\left[\begin{array}{ccc}
\ell_{0} / \ell & 0 & 0 \\
0 & a_{0} / a & 0 \\
0 & 0 & b_{0} / b
\end{array}\right]
\end{array}\right.
$$

The state of stress is uniaxial with the $x$-component the only nonzero component, so

$$
\sigma=\left[\begin{array}{ccc}
\sigma_{x} & 0 & 0  \tag{E3.9.5}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

Evaluating $\mathbf{P}$ as given by Box 3.2 using Eqs. (E3.9.3-E3.9.5) then gives

$$
\left.\left.\left.\mathbf{P}=\frac{a b \ell}{a_{0} b_{0} \ell_{0}} \left\lvert\, \begin{array}{ccc}
\left\lceil\ell_{0} / \ell\right. & 0 & 0  \tag{E3.9.6}\\
0 & a_{0} / a & 0 \\
0 & 0 & b_{0} / b
\end{array}\right.\right] \begin{array}{ccc}
\| \sigma_{x} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]=\left\lvert\, \begin{array}{ccc}
\frac{a b \sigma_{x}}{a_{0} b_{0}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right.\right]
$$

Thus the only nonzero component of the nominal stress is

$$
\begin{equation*}
P_{11}=\frac{a b}{a_{0} b_{0}} \sigma_{x}=\frac{A \sigma_{x}}{A_{0}} \tag{E3.9.7}
\end{equation*}
$$

where the last equality is based on the formulas for the cross-sectional area, $A=a b$ and $A_{0}=a_{0} b_{0}$; Eq. (E3.9.7) agrees with Eq. (2.2.7). Thus, in a state of uniaxial stress, $P_{11}$ corresponds to the engineering stress.

The relationship between the PK2 stress and Cauchy stress for a uniaxial state of stress is obtained by using Eqs. (E3.9.3-E3.9.5) with Eq. (3.4.14), which gives

$$
\begin{equation*}
S_{11}=\frac{\ell_{0}}{\ell}\left(\frac{A \sigma_{x}}{A_{o}}\right) \tag{E3.9.8}
\end{equation*}
$$

where the quantity in the parenthesis can be recognized as the nominal stress. It can be seen from the above that it is difficult to ascribe a physical meaning to the PK2 stress. This, as will be seen in Chapter 5, influences the selection of stress measures for plasticity theories, since yield functions must be described in terms of physical stresses. Because of the nonphysical nature of the nominal and PK2 stresses, it is awkward to formulate plasticity in terms of these stresses.

### 3.5 CONSERVATION EQUATIONS

3.5.1 Conservation Laws. One group of the fundamental equations of continuum mechanics arises from the conservation laws. These equations must always be satisfied by physical systems. Four conservation laws relevant to thermomechanical systems are considered here:

1. conservation of mass
2. conservation of linear momentum, often called conservation of momentum
3. conservation of energy
4. conservation of angular momentum

The conservation laws are also known as balance laws, e.g. the conservation of energy is often called the balance of energy.

The conservation laws are usually expressed as partial differential equations (PDEs). These PDEs are derived by applying the conservation laws to a domain of the body, which leads to an integral equation. The following relationship is used to extract the PDEs from the integral equation:

$$
\text { if } f(\mathbf{x}, t) \text { is } C^{-1} \text { and } \int_{\Omega} f(\mathbf{x}, t) d \Omega=0 \text { for any subdomain } \Omega \text { of } \bar{\Omega}
$$

and time $t \in[0, \bar{t}]$, then

$$
\begin{equation*}
f(\mathbf{x}, t)=0 \text { in } \Omega \text { for } t \in[0, \bar{t}] \tag{3.5.1}
\end{equation*}
$$

In the following, $\Omega$ is an arbitrary subdomain of the body under consideration. Prior to deriving the balance equations, several theorems useful for this purpose are derived.
3.5.2 Gauss's Theorem. In the derivation of the governing equations, Gauss's theorem is frequently used. This theorem relates integrals of different dimensions: it can be used to relate a contour integral to an area integral or a surface integral to a volume integral. The one dimensional form of Gauss's theorem is the fundamental theorem of calculus, which we used in Chapter 2.

Gauss's theorem states that when $f(\mathbf{x})$ is a piecewise continuously diffrentiable, i.e. $C^{1}$ function, then

$$
\begin{align*}
& \int_{\Omega} \frac{\partial f(\mathbf{x})}{\partial x_{i}} d \Omega=\int_{\Gamma} f(\mathbf{x}) n_{i} d \Gamma \quad \text { or } \quad \int_{\Omega} \nabla f(\mathbf{x}) d \Omega=\int_{\Gamma} f(\mathbf{x}) \mathbf{n} d \Gamma  \tag{3.5.2a}\\
& \int_{\Omega_{0}} \frac{\partial f(\mathbf{X})}{\partial X_{i}} d \Omega_{0}=\int_{\Gamma_{0}} f(\mathbf{X}) n_{i}^{0} d \Gamma_{0} \quad \text { or } \int_{\Omega_{0}} \nabla_{\mathbf{X}} f(\mathbf{X}) d \Omega_{0}=\int_{\Gamma_{0}} f(\mathbf{X}) \mathbf{n}_{0} d \Gamma_{0} \tag{3.5.2b}
\end{align*}
$$

As seen in the above, Gauss's theorem applies to integrals in both the current and reference configurations.

The above theorem holds for a tensor of any order; for example if $f(\mathbf{x})$ is replaced by a tensor of first order, then

$$
\begin{equation*}
\int_{\Omega} \frac{\partial g_{i}(\mathbf{x})}{\partial x_{i}} d \Omega=\int_{\Gamma} g_{i}(\mathbf{x}) n_{i} d \Gamma \quad \text { or } \quad \int_{\Omega} \nabla \cdot \mathbf{g}(\mathbf{x}) d \Omega=\int_{\Gamma} \mathbf{n} \cdot \mathbf{g}(\mathbf{x}) d \Gamma \tag{3.5.3}
\end{equation*}
$$

which is often known as the divergence theorem. The theorem also holds for gradients of the vector field:

$$
\begin{equation*}
\int_{\Omega} \frac{\partial g_{i}(\mathbf{x})}{\partial x_{j}} d \Omega=\int_{\Gamma} g_{i}(\mathbf{x}) n_{j} d \Gamma \quad \text { or } \quad \int_{\Omega} \nabla \mathbf{g}(\mathbf{x}) d \Omega=\int_{\Gamma} \mathbf{n} \otimes \mathbf{g}(\mathbf{x}) d \Gamma \tag{3.5.3b}
\end{equation*}
$$

and to tensors of arbitrary order.
If the function $f(\mathbf{x})$ is not continuously differentiable, i.e. if its derivatives are discontinuous along a finite number of lines in two dimensions or on surfaces in three dimensions, then $\Omega$ must be subdivided into subdomains so that the function is $C^{1}$ within each subdomain. Discontinuities in the derivatives of the function will then occur only on the interfaces between the subdomains. Gauss's theorem is applied to each of the subdomains, and summing the results yields the following counterparts of (3.5.2) and (3.5.3):

$$
\begin{equation*}
\int_{\Omega} \frac{\partial f}{\partial x_{i}} d \Omega=\int_{\Gamma} f n_{i} d \Gamma+\int_{\Gamma_{\text {int }}}\left\langle f n_{i}\right\rangle d \Gamma \quad \int_{\Omega} \frac{\partial g_{i}}{\partial x_{i}} d \Omega=\int_{\Gamma} g_{i} n_{i} d \Gamma+\int_{\Gamma_{\text {int }}}\left\langle g_{i} n_{i}\right\rangle d \Gamma \tag{3.5.4}
\end{equation*}
$$

where $\Gamma_{i n t}$ is the set of all interfaces between these subdomains and $\{f\rangle$ and $\langle\mathbf{n} \cdot \mathbf{g}\rangle$ are the jumps defined by

$$
\begin{align*}
& \langle f\rangle=f^{A}-f^{B}  \tag{3.5.5a}\\
& \langle\mathbf{n} \cdot \mathbf{g}\rangle=\left\langle g_{i} n_{i}\right\rangle=g_{i}^{A} n_{i}^{A}+g_{i}^{B} n_{i}^{B}=\left(g_{i}^{A}-g_{i}^{B}\right) n_{i}^{A}=\left(g_{i}^{B}-g_{i}^{A}\right) n_{i}^{B} \tag{3.5.5b}
\end{align*}
$$

where $A$ and $B$ are a pair of subdomains which border on the interface $\Gamma_{i n t}, \mathbf{n}^{A}$ and $\mathbf{n}^{B}$ are the outward normals for the two subdomains and $f^{A}$ and $f^{B}$ are the function values at the points adjacent to the interface in subdomains $A$ and $B$, respectively. All the forms in (3.5.5b) are equivalent and make use of the fact that on the interface, $\mathbf{n}^{A}=-\mathbf{n}^{B}$. The first of the formulas is the easiest to remember because of its symmetry with respect to $A$ and $B$.

### 3.5.3 Material Time Derivative of an Integral and Reynold's

 Transport Theorem. The material time derivative of an integral is the rate of change of an integral on a material domain. A material domain moves with the material, so that the material points on the boundary remain on the boundary and no flux occurs across the boundaries. A material domain is analogous to a Lagrangian mesh; a Lagrangian element or group of Lagrangian elements is a nice example of a material domain. The various forms for material time derivatives of integrals are called Reynold; s transport theorem, which is employed in the development of conservation laws.The material time derivative of an integral is defined by

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f d \Omega=\lim _{\Delta t \rightarrow 0} \frac{1}{\Delta t}\left(\int_{\Omega_{\tau+\Delta t}} f(\mathbf{x}, \tau+\Delta t) d \Omega_{\mathbf{x}}-\int_{\Omega_{\tau}} f(\mathbf{x}, \tau) d \Omega_{\mathbf{x}}\right) \tag{3.5.6}
\end{equation*}
$$

where $\Omega_{\tau}$ is the spatial domain at time $\tau$ and $\Omega_{\tau+\Delta t}$ the spatial domain occupied by the same material points at time $\tau+\Delta t$. The notation on the left hand side is a little confusing because it appears to refer to a single spatial domain. However, in this notation, which is standard, the material derivative on the integral implies that the domain refers to a material domain. We now transform both integrals on the right hand side to the reference domain using (3.2.18) and change the independent variables to the material coordinates, which gives

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f d \Omega=\lim _{\Delta t \rightarrow 0} \frac{1}{\Delta t}\left(\int_{\Omega_{0}} f(\mathbf{X}, \tau+\Delta t) J(\mathbf{X}, \tau+\Delta t) d \Omega_{0}-\int_{\Omega_{0}} f(\mathbf{X}, \tau) J(\mathbf{X}, \tau) d \Omega_{0}\right) \tag{3.5.7}
\end{equation*}
$$

The function is now $f(\phi(\mathbf{X}, t), t) \equiv f \circ \phi$, but we adhere to our convention that the symbol represents the field and leave the symbol unchanged.

Since the domain of integration is now independent of time, we can pull the limit operation inside the integral and take the limit, which yields

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f d \Omega=\int_{\Omega_{0}} \frac{\partial}{\partial t}(f(\mathbf{X}, t) J(\mathbf{X}, t)) d \Omega_{0} \tag{3.5.9}
\end{equation*}
$$

The partial derivative with respect to time in the integrand is a material time derivative since the independent space variables are the material coordinates. We next use the product rule for derivatives on the above:

$$
\frac{D}{D t} \int_{\Omega} f d \Omega=\int_{\Omega_{0}} \frac{\partial}{\partial t}(f(\mathbf{X}, t) J(\mathbf{X}, t)) d \Omega_{0}=\int_{\Omega_{0}}\left(\frac{\partial f}{\partial t} J+f \frac{\partial J}{\partial t}\right) d \Omega_{0}
$$

Bearing in mind that the partial time derivatives are material time derivatives because the independent variables are the material coordinates and time, we can use (3.2.19) to obtain

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f d \Omega=\int_{\Omega_{0}}\left(\frac{\partial f(\mathbf{X}, t)}{\partial t} J+f J \frac{\partial v_{i}}{\partial x_{i}}\right) d \Omega_{0} \tag{3.5.12}
\end{equation*}
$$

We can now transform the RHS integral to the current domain using (3.2.18) and change the independent variables to an Eulerian description, which gives

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f(\mathbf{x}, t) d \Omega=\int_{\Omega}\left(\frac{D f(\mathbf{x}, t)}{D t}+f \frac{\partial v_{i}}{\partial x_{i}}\right) d \Omega \tag{3.5.11}
\end{equation*}
$$

where the partial time derivative has been changed to a material time derivative because of the change of independent variables; the material time derivative symbol has been changed with the change of independent variables, since $D f(\mathbf{x}, t) / D t \equiv \partial f(\mathbf{X}, t) / \partial t$ as indicated in (3.2.8).

An alternate form of Reynold's transport theorem can be obtained by using the definition of the material time derivative, Eq. (3.2.12) in (3.5.11). This gives

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f d \Omega=\int_{\Omega}\left(\frac{\partial f}{\partial t}+v_{i} \frac{\partial f}{\partial x_{i}}+\frac{\partial v_{i}}{\partial x_{i}} f\right) d \Omega=\int_{\Omega}\left(\frac{\partial f}{\partial t}+\frac{\partial\left(v_{i} f\right)}{\partial x_{i}}\right) d \Omega \tag{3.5.13}
\end{equation*}
$$

which can be written in tensor form as

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f d \Omega=\int_{\Omega}\left(\frac{\partial f}{\partial t}+\operatorname{div}(\mathbf{v} f)\right) d \Omega \tag{3.5.14}
\end{equation*}
$$

Equation (3.5.14) can be put into another form by using Gauss's theorem on the second term of the RHS, which gives

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} f d \Omega=\int_{\Omega} \frac{\partial f}{\partial t} d \Omega+\int_{\Gamma} f v_{i} n_{i} d \Gamma \quad \text { or } \quad \frac{D}{D t} \int_{\Omega} f d \Omega=\int_{\Omega} \frac{\partial f}{\partial t} d \Omega+\int_{\Gamma} f \mathbf{v} \cdot \mathbf{n} d \Gamma \tag{3.5.15}
\end{equation*}
$$

where the product $f \mathbf{v}$ is assumed to be $C^{1}$ in $\Omega$. Reynold's transport theorem, which in the above has been given for a scalar, applies to a tensor of any order. Thus to apply it to a first order tensor (vector) $g_{k}$, replace $f$ by $g_{k}$ in Eq. (3.5.14), which gives

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} g_{k} d \Omega=\int_{\Omega}\left(\frac{\partial g_{k}}{\partial t}+\frac{\partial\left(v_{i} g_{k}\right)}{\partial x_{i}}\right) d \Omega \tag{3.5.16}
\end{equation*}
$$

3.5.5 Mass Conservation. The mass $m(\Omega)$ of a material domain $\Omega$ is given by

$$
\begin{equation*}
m(\Omega)=\int_{\Omega} \rho(\mathbf{X}, t) d \Omega \tag{3.5.17}
\end{equation*}
$$

where $\rho(\mathbf{X}, t)$ is the density. Mass conservation requires that the mass of a material subdomain be constant, since no material flows through the boundaries of a material subdomain and we are not considering mass to energy conversion. Therefore, according to the principle of mass conservation, the material time derivative of $m(\Omega)$ vanishes, i.e.

$$
\begin{equation*}
\frac{D m}{D t}=\frac{D}{D t} \int_{\Omega} \rho d \Omega=0 \tag{3.5.18}
\end{equation*}
$$

Applying Reynold's theorem, Eq. (3.5.11), to the above yields

$$
\begin{equation*}
\int_{\Omega}\left(\frac{D \rho}{D t}+\rho \operatorname{div}(\mathbf{v})\right) d \Omega=0 \tag{3.5.19}
\end{equation*}
$$

Since the above holds for any subdomain $\Omega$, it follows from Eq.(3.5.1) that

$$
\begin{equation*}
\frac{D \rho}{D t}+\rho \operatorname{div}(\mathbf{v})=0 \quad \text { or } \quad \frac{D \rho}{D t}+\rho v_{i, i}=0 \quad \text { or } \quad \dot{\rho}+\rho v_{i, i}=0 \tag{3.5.20}
\end{equation*}
$$

The above is the equation of mass conservation, often called the continuity equation. It is a first order partial differential equation.

Several special forms of the mass conservation equation are of interest. When a material is incompressible, the material time derivative of the density vanishes, and it can be seen from equation (3.5.20) that the mass conservation equation becomes:

$$
\begin{equation*}
\operatorname{div}(\mathbf{v})=0 \quad v_{i, i}=0 \tag{3.5.21}
\end{equation*}
$$

In other words, mass conservation requires the divergence of the velocity field of an incompressible material to vanish.

If the definition of a material time derivative, (3.2.12) is invoked in (3.5.20), then the continuity equation can be written in the form

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\rho_{, i} v_{i}+\rho v_{i, i}=\frac{\partial \rho}{\partial t}+\left(\rho v_{i}\right)_{, i}=0 \tag{3.5.22}
\end{equation*}
$$

This is called the conservative form of the mass conservation equation. It is often preferred in computational fluid dynamics because discretizations of the above form are thouught to more accurately enforce mass conservation.

For Lagrangian descriptions, the rate form of the mass conservation equation, Eq. (3.5.18), can be integrated in time to obtain an algebraic equation for the density. Integrating Eq. (3.5.18) in time gives

$$
\begin{equation*}
\int_{\Omega} \rho d \Omega=\text { constant }=\int_{\Omega_{0}} \rho_{0} d \Omega_{0} \tag{3.5.23}
\end{equation*}
$$

Transforming the left hand integral in the above to the reference domain by (3.2.18) gives

$$
\begin{equation*}
\int_{\Omega_{0}}\left(\rho J-\rho_{0}\right) d \Omega_{\mathrm{o}}=0 \tag{3.5.24}
\end{equation*}
$$

Then invoking the smoothness of the integrand and Eq. (3.5.1) gives the following equation for mass conservation

$$
\begin{equation*}
\rho(\mathbf{X}, t) J(\mathbf{X}, t)=\rho_{0}(\mathbf{X}) \quad \text { or } \quad \rho J=\rho_{0} \tag{3.5.25}
\end{equation*}
$$

We have explicitly indicated the independent variables on the left to emphasize that this equation only holds for material points; the fact that the independent variables must be the material coordinates in these equations follows from the fact that the integrand and domain of integration in (3.5.24) must be expressed for a material coordinate and material subdomain, respectively.

As a consequence of the integrability of the mass conservation equation in Lagrangian descriptions, the algebraic equation (3.5.25) are used to enforce mass conservation in Lagrangian meshes. In Eulerian meshes, the algebraic form of mass conservation, Eq. (3.5.25), cannot be used, and mass conservation is imposed by the partial differential equation, (3.5.20) or (3.5.22), i.e. the continuity equation.
3.5.5 Conservation of Linear Momentum. The equation emanating from the principle of momentum conservation is a key equation in nonlinear finite element procedures. Momentum conservation is a statement of Newton's second law of motion, which relates the forces acting on a body to its acceleration. We consider an arbitrary subdomain of the body $\Omega$ with boundary $\Gamma$. The body is subjected to body forces $\rho \mathbf{b}$ and to surface tractions $\mathbf{t}$, where $\mathbf{b}$ is a force per unit mass and $\mathbf{t}$ is a force per unit area. The total force on the body is given by

$$
\begin{equation*}
\mathbf{f}(t)=\int_{\Omega} \rho \mathbf{b}(\mathbf{x}, t) d \Omega+\int_{\Gamma} \mathbf{t}(\mathbf{x}, t) d \Gamma \tag{3.5.26}
\end{equation*}
$$

The linear momentum of the body is given by

$$
\begin{equation*}
\mathbf{p}(t)=\int_{\Omega} \rho \mathbf{v}(\mathbf{x}, t) d \Omega \tag{3.5.27}
\end{equation*}
$$

where $\rho \mathbf{v}$ is the linear momentum per unit volume.

Newton's second law of motion for a continuum states that the material time derivative of the linear momentum equals the net force. Using (3.5.26) and (3.5.27), this gives

$$
\begin{equation*}
\frac{D \mathbf{p}}{D t}=\mathbf{f} \Rightarrow \frac{D}{D t} \int_{\Omega} \rho \mathbf{v} d \Omega=\int_{\Omega} \rho \mathbf{b} d \Omega+\int_{\Gamma} \mathbf{t} d \Gamma \tag{3.5.28}
\end{equation*}
$$

We now convert the first and third integrals in the above to obtain a single domain integral so Eq. (3.5.1) can be applied. Reynold's Transport Theorem applied to the first integral in the above gives

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} \rho \mathbf{v} d \Omega=\int_{\Omega}\left(\frac{D}{D t}(\rho \mathbf{v})+\operatorname{div}(\mathbf{v}) \rho \mathbf{v}\right) d \Omega=\int_{\Omega}\left(\rho \frac{D \mathbf{v}}{D t}+\mathbf{v}\left(\frac{D \rho}{D t}+\rho d i v(\mathbf{v})\right) d \Omega\right. \tag{3.5.29}
\end{equation*}
$$

where the second equality is obtained by using the product rule of derivatives for the first term of the integrand and rearranging terms.

The term multiplying the velocity in the RHS of the above can be recognized as the continuity equation, which vanishes, giving

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} \rho \mathbf{v} d \Omega=\int_{\Omega} \rho \frac{D \mathbf{v}}{D t} d \Omega \tag{3.5.30}
\end{equation*}
$$

To convert the last term in Eq. (3.5.28) to a domain integral, we invoke Cauchy's relation and Gauss's theorem in sequence, giving

$$
\begin{equation*}
\int_{\Gamma} \mathbf{t} d \Gamma=\int_{\Gamma} \mathbf{n} \cdot \sigma d \Gamma=\int_{\Omega} \nabla \cdot \sigma d \Omega \quad \text { or } \quad \int_{\Gamma} t_{j} d \Gamma=\int_{\Gamma} n_{i} \sigma_{i j} d \Gamma=\int_{\Omega} \frac{\partial \sigma_{i j}}{\partial x_{i}} d \Omega \tag{3.5.31}
\end{equation*}
$$

Note that since the normal is to the left on the boundary integral, the divergence is to the left and contracts with the first index on the stress tensor. When the divergence operator acts on the first index of the stress tensor it is called the left divergence operator and is placed to the left of operand. When it acts on the second index, it is placed to the right and call the right divergence. Since the Cauchy stress is symmetric, the left and right divergence operators have the same effect. However, in contrast to linear continuum mechanics, in nonlinear continuum mechanics it is important to become accustomed to placing the divergence operator where it belongs because some stress tensors, such as the nominal stress, are not symmetric. When the stress is not symmetric, the left and right divergence operators lead to different results. When Gauss's theorem is used, the divergence on the stress tensor is on the same side as the normal in Cauchy's relation. In this book we will use the convention that the normal and divergence are always placed on the left.

Substituting (3.5.30) and (3.5.31) into (3.5.28) gives

$$
\begin{equation*}
\int_{\Omega}\left(\rho \frac{D \mathbf{v}}{D t}-\rho \mathbf{b}-\nabla \cdot \sigma\right) d \Omega=0 \tag{3.5.32}
\end{equation*}
$$

Therefore, if the integrand is $\mathrm{C}^{-1}$, since (3.5.32) holds for an arbitrary domain, applying (3.5.1) yields

$$
\begin{equation*}
\rho \frac{D \mathbf{v}}{D t}=\nabla \cdot \sigma+\rho \mathbf{b} \equiv d i v \sigma+\rho \mathbf{b} \quad \text { or } \quad \rho \frac{D v_{i}}{D t}=\frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i} \tag{3.5.33}
\end{equation*}
$$

This is called the momentum equation or the equation of motion; it is also called the balance of linear momentum equation. The LHS term represents the change in momentum, since it is a product of the acceleration and the density; it is also called the inertial term. The first term on the RHS is the net resultant internal force per unit volume due to divergence of the stress field.

This form of the momentum equation is applicable to both Lagrangian and Eulerian descriptions. In a Lagrangian description, the dependent variables are assumed to be functions of the Lagrangian coordinates $\mathbf{X}$ and time $t$, so the momentum equation is

$$
\begin{equation*}
\rho(\mathbf{X}, t) \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}=\operatorname{div} \sigma\left(\phi^{-1}(\mathbf{x}, t), t\right)+\rho(\mathbf{X}, t) \mathbf{b}(\mathbf{X}, t) \tag{3.5.34}
\end{equation*}
$$

Note that the stress must be expressed as a function of the Eulerian coordinates through the motion $\phi^{-1}(\mathbf{X}, t)$ so that the spatial divergence of the stress field can be evaluated; the total derivative of the velocity with respect to time in (3.5.33) becomes a partial derivative with respect to time when the independent variables are changed from the Eulerian coordinates $\mathbf{x}$ to the Lagrangian coordinates $\mathbf{X}$.

In an Eulerian description, the material derivative of the velocity is written out by (3.2.9) and all variables are considered functions of the Eulerian coordinates. Equation (3.5.33) becomes

$$
\begin{align*}
& \rho(\mathbf{x}, t)\left(\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t}+(\mathbf{v}(\mathbf{x}, t) \cdot \operatorname{grad}) \mathbf{v}(\mathbf{x}, t)\right)=\operatorname{div} \sigma(\mathbf{x}, t)+\rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t)  \tag{3.5.35}\\
& \quad \text { or } \rho\left(\frac{\partial v_{i}}{\partial t}+v_{i, j} v_{j}\right)=\frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i}
\end{align*}
$$

As can be seen from the above, when the independent variables are all explicitly written out the equations are quite awkward, so we will usually drop the independent variables. The independent variables are specified wherever the dependent variables are first defined, when they first appear in a section or chapter, or when they are changed. So if the independent variables are not clear, the reader should look back to where the independent variables were last specified.

In computational fluid dynamics, the momentum equation is sometimes used without the changes made by Eqs. (3.5.13-3.5.30). The resulting equation is

$$
\begin{equation*}
\frac{D(\rho \mathbf{v})}{D t} \equiv \frac{\partial(\rho \mathbf{v})}{\partial t}+\mathbf{v} \cdot \operatorname{grad}(\rho \mathbf{v})=\operatorname{div} \sigma+\rho \mathbf{b} \tag{3.5.36}
\end{equation*}
$$

This is called the conservative form of the momentum equation with considered $\rho \mathbf{v}$ as one of the unknowns. Treating the equation in this form leads to better observance of momentum conservation.
3.5.7 Equilibrium Equation. In many problems, the loads are applied slowly and the inertial forces are very small and can be neglected. In that case, the acceleration in the momentum equation (3.5.35) can be dropped and we have

$$
\begin{equation*}
\nabla \cdot \sigma+\rho \mathbf{b}=0 \quad \text { or } \frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i}=0 \tag{3.5.37}
\end{equation*}
$$

The above equation is called the equilibrium equation. Problems to which the equilibrium equation is applicable are often called static problems. The equilibrium equation should be carefully distinguished from the momentum equation: equilibrium processes are static and do not include acceleration. The momentum and equilibrium equations are tensor equations, and the tensor forms (3.5.33) and (3.5.37) represent $n_{S D}$ scalar equations.
3.5.8 Reynold's Theorem for a Density-Weighted Integrand. Equation (3.5.30) is a special case of a general result: the material time derivative of an integral in which the integrand is a product of the density and the function $f$ is given by

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} \rho f d \Omega=\int_{\Omega} \rho \frac{D f}{D t} d \Omega \tag{3.5.38}
\end{equation*}
$$

This holds for a tensor of any order and is a consequence of Reynold's theorem and mass conservation; thus, it can be called another form of Reynold's theorem. It can be verified by repeating the steps in Eqs. (3.5.29) to (3.5.30) with a tensor of any order.
3.5.9 Conservation of Angular Momentum. The conservation of angular momentum provides additional equations which govern the stress tensors. The integral form of the conservation of angular momentum is obtained by taking the cross-product of each term in the corresponding linear momentum principle with the position vector $\mathbf{x}$, giving

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} \mathbf{x} \times \rho \mathbf{v} d \Omega=\int_{\Omega} \mathbf{x} \times \rho \mathbf{b} d \Omega+\int_{\Gamma} \mathbf{x} \times \mathbf{t} d \Gamma \tag{3.5.39}
\end{equation*}
$$

We will leave the derivation of the conditions which follow from (3.5.39) as an exercise and only state them:

$$
\begin{equation*}
\sigma=\sigma^{T} \quad \text { or } \quad \sigma_{i j}=\sigma_{j i} \tag{3.5.40}
\end{equation*}
$$

In other words, conservation of angular momentum requires that the Cauchy stress be a symmetric tensor. Therefore, the Cauchy stress tensor represents 3 distinct dependent variables in two-dimensional problems, 6 in three-dimensional problems. The conservation of angular momentum does not result in any additional partial differential equations when the Cauchy stress is used.
3.5.10 Conservation of Energy. We consider thermomechanical processes where the only sources of energy are mechanical work and heat. The principle of conservation of energy, i.e. the energy balance principle, states that the rate of change of total energy is equal to the work done by the body forces and surface tractions plus the heat energy delivered to the body by the heat flux and other sources of heat. The internal energy per unit volume is denoted by $\rho w^{\text {int }}$ where $w^{i n t}$ is the internal energy per unit mass. The heat flux per unit area is denoted by a vector $\mathbf{q}$, in units of power per area and the heat source per unit volume is denoted by $\rho s$. The conservation of energy then requires that the rate of change of the total energy in the body, which includes both internal energy and kinetic energy, equal the power of the applied forces and the energy added to the body by heat conduction and any heat sources.

The rate of change of the total energy in the body is given by

$$
\begin{equation*}
\mathcal{P}^{t o t}=\mathcal{P}^{i n t}+\mathcal{P}^{k i n}, \quad \mathcal{P}^{i n t}=\frac{D}{D t} \int_{\Omega} \rho w^{i n t} d \Omega, \quad P^{k i n}=\frac{D}{D t} \int_{\Omega} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} d \Omega \tag{3.5.41}
\end{equation*}
$$

where $\mathcal{P}^{\text {int }}$ denotes the rate of change of internal energy and $\mathscr{P}^{k i n}$ the rate of change of the kinetic energy. The rate of the work by the body forces in the domain and the tractions on the surface is

$$
\begin{equation*}
\mathcal{P}^{e x t}=\int_{\Omega} \mathbf{v} \cdot \rho \mathbf{b} d \Omega+\int_{\Gamma} \mathbf{v} \cdot \mathbf{t} d \Gamma=\int_{\Omega} v_{i} \rho b_{i} d \Omega+\int_{\Gamma} v t_{i} d \Gamma \tag{3.5.42}
\end{equation*}
$$

The power supplied by heat sources $s$ and the heat flux $\mathbf{q}$ is

$$
\begin{equation*}
\mathcal{P}^{\text {heat }}=\int_{\Omega} \rho s d \Omega-\int_{\Gamma} \mathbf{n} \cdot \mathbf{q} d \Gamma=\int_{\Omega} \rho s d \Omega-\int_{\Gamma} n_{i} q_{i} d \Gamma \tag{3.5.43}
\end{equation*}
$$

where the sign of the heat flux term is negative since positive heat flow is out of the body.

The statement of the conservation of energy is written

$$
\begin{equation*}
P^{t o t}=P^{e x t}+P^{h e a t} \tag{3.5.44}
\end{equation*}
$$

i.e. the rate of change of the total energy in the body (consisting of the internal and kinetic energies) is equal to the rate of work by the external forces and rate of work provided by heat flux and energy sources. This is known as the first law of thermodynamics. The disposition of the internal work depends on the material. In an elastic material, it is stored as elastic internal energy and fully recoverable upon unloading. In an elastic-plastic material, some of it is converted to heat, whereas some of the energy is irretrievably dissipated in changes of the internal structure of the material.

Substituting Eqs. (3.5.41) to (3.5.43) into (3.5.44) gives the full statement of the conservation of energy

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega}\left(\rho w^{\text {int }}+\frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v}\right) d \Omega=\int_{\Omega} \mathbf{v} \cdot \rho \mathbf{b} d \Omega+\int_{\Gamma} \mathbf{v} \cdot \mathbf{t} d \Gamma+\int_{\Omega} \rho s d \Omega-\int_{\Gamma} \mathbf{n} \cdot \mathbf{q} d \Gamma \tag{3.5.45}
\end{equation*}
$$

We will now derive the equation which emerges from the above integral statement using the same procedure as before: we use Reynolds's theorem to bring the total derivative inside the integral and convert all surface integrals to domain integrals. Using Reynold's Theorem, (3.5.38) on the first integral in Eq. (3.5.45) gives

$$
\begin{align*}
\frac{D}{D t} \int_{\Omega}\left(\rho w^{i n t}+\frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v}\right) d \Omega= & \int_{\Omega}\left(\rho \frac{D w^{i n t}}{D t}+\frac{1}{2} \rho \frac{D(\mathbf{v} \cdot \mathbf{v})}{D t}\right) d \Omega \\
& =\int_{\Omega}\left(\rho \frac{D w^{i n t}}{D t}+\rho \mathbf{v} \cdot \frac{D \mathbf{v}}{D t}\right) d \Omega \tag{3.5.46}
\end{align*}
$$

We will use commas in the following to denote spatial derivatives. Applying Cauchy's law (3.4.1) and Gauss's theorem (3.5.12) to the traction boundary integrals on the RHS of (3.5.45) yields:

$$
\begin{align*}
\int_{\Gamma} \mathbf{v} \cdot \mathbf{t} d \Gamma & =\int_{\Gamma} \mathbf{n} \cdot \sigma \cdot \mathbf{v} d \Gamma=\int_{\Omega}\left(v_{i} \sigma_{j i}\right)_{j} d \Omega=\int_{\Omega}\left(v_{i, j} \sigma_{j i}+v_{i} \sigma_{j i, j}\right) d \Omega \\
& =\int_{\Omega}\left(D_{j i} \sigma_{j i}+W_{j i} \sigma_{j i}+v_{i} \sigma_{j i, j}\right) d \Omega \\
& =\int_{\Omega}\left(D_{j i} \sigma_{j i}+v_{i} \sigma_{j i, j}\right) d \Omega \quad \\
& =\int_{\Omega}(\mathbf{D}: \sigma+(\nabla \cdot \sigma) \cdot \mathbf{v}) d \Omega \tag{3.5.47}
\end{align*}
$$

Inserting these results into (3.5.44) or (3.5.45), application of Gauss's theorem to the heat flux integral and rearrangement of terms yields

$$
\begin{equation*}
\int_{\Omega}\left(\rho \frac{D w^{\text {int }}}{D t}-\mathbf{D}: \sigma+\nabla \cdot \mathbf{q}-\rho s+\mathbf{v} \cdot\left(\rho \frac{D \mathbf{v}}{D t}-\nabla \cdot \sigma-\rho \mathbf{b}\right)\right) d \Omega=0 \tag{3.5.48}
\end{equation*}
$$

The last term in the integral can be recognized as the momentum equation, Eq. (3.5.33), so it vanishes. Then invoking the arbitrariness of the domain gives:

$$
\begin{equation*}
\rho \frac{D w^{i n t}}{D t}=\mathbf{D}: \sigma-\nabla \cdot \mathbf{q}+\rho s \tag{3.5.49}
\end{equation*}
$$

When the heat flux and heat sources vanish, i.e. in a purely mechanical process, the energy equation becomes

$$
\begin{equation*}
\rho \frac{D w^{i n t}}{D t}=\mathbf{D}: \sigma=\sigma: \mathbf{D}=\sigma_{i j} D_{i j} \tag{3.5.50}
\end{equation*}
$$

The above defines the rate of internal energy or internal power in terms of the measures of stress and strain. It shows that the internal power is given by the contraction of the rate-of-deformation and the Cauchy stress. We therefore say that the rate-of-deformation and the Cauchy stress are conjugate in power. As we shall see, conjugacy in power is helpful in the development of weak forms: measures of stress and strain rate which are conjugate in power can be used to construct principles of virtual work or power, which are the weak forms for finite element approximations of the momentum equation. Variables which are conjugate in power are also said to be conjugate in work or energy, but we will use the phrase conjugate in power because it is more accurate.

The rate of change of the internal energy of the system is obtained by integrating (3.5.50) over the domain of the body, which gives

$$
\begin{equation*}
\frac{D W^{i n t}}{D t}=\int_{\Omega} \rho \frac{D w^{i n t}}{D t} d \Omega=\int_{\Omega} \mathbf{D}: \sigma d \Omega=\int_{\Omega} D_{i j} \sigma_{i j} d \Omega=\int_{\Omega} \frac{\partial v_{i}}{\partial x_{j}} \sigma_{i j} d \Omega \tag{3.5.51}
\end{equation*}
$$

where the last expression follows from the symmetry of the Cauchy stress tensor.
The conservation equations are summarized in Box 3.3 in both tensor and indicial form. The equations are written without specifying the independent variables; they can be expressed in terms of either the spatial coordinates or the material coordinates, and as we shall see later, they can be written in terms of other coordinate systems which are neither fixed in space nor coincident with material points. The equations are not expressed in conservative form because this does not seem to be as useful in solid mechanics as it is in fluid mechanics. The reasons for this are not explored in the literature, but it appears to be related to the mauch smaller changes in density which occur in solid mechanics problems.

## Box 3.3 Conservation Equations

## Eulerian description

Mass

$$
\begin{equation*}
\frac{D \rho}{D t}+\rho \operatorname{div}(\mathbf{v})=0 \quad \text { or } \quad \frac{D \rho}{D t}+\rho v_{i, i}=0 \quad \text { or } \quad \dot{\rho}+\rho v_{i, i}=0 \tag{B3.3.1}
\end{equation*}
$$

Linear Motion

$$
\begin{equation*}
\rho \frac{D \mathbf{v}}{D t}=\nabla \cdot \sigma+\rho \mathbf{b} \equiv d i v \sigma+\rho \mathbf{b} \quad \text { or } \quad \rho \frac{D v_{i}}{D t}=\frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i} \tag{B3.3.2}
\end{equation*}
$$

Angular Momentum

$$
\begin{equation*}
\sigma=\sigma^{T} \quad \text { or } \quad \sigma_{i j}=\sigma_{j i} \tag{B3.3.3}
\end{equation*}
$$

Energy

$$
\begin{equation*}
\rho \frac{D w^{\text {int }}}{D t}=\mathbf{D}: \sigma-\nabla \cdot \mathbf{q}+\rho s \tag{B3.3.4}
\end{equation*}
$$

Lagrangian Description
Mass

$$
\begin{equation*}
\rho(\mathbf{X}, t) J(\mathbf{X}, t)=\rho_{0}(\mathbf{X}) \quad \text { or } \quad \rho J=\rho_{0} \tag{B3.3.5}
\end{equation*}
$$

Linear Momentum

$$
\begin{equation*}
\rho_{0} \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}=\nabla_{\mathbf{X}} \cdot \mathbf{P}+\rho_{0} \mathbf{b} \quad \text { or } \quad \rho_{0} \frac{\partial v_{i}(\mathbf{X}, t)}{\partial t}=\frac{\partial P_{j i}}{\partial X_{j}}+\rho_{0} b_{i} \tag{B3.3.6}
\end{equation*}
$$

Angular Momentum

$$
\begin{align*}
& \quad \mathbf{F} \cdot \mathbf{P}=\mathbf{P}^{T} \cdot \mathbf{F}^{T} \quad F_{i k} P_{k j}=P_{i k}^{T} F_{k j}^{T}=F_{j k} P_{k i} \\
& \quad \mathbf{S}=\mathbf{S}^{T}  \tag{B3.3.7}\\
& \text { Energy } \tag{B3.3.8}
\end{align*}
$$

$$
\begin{equation*}
\rho_{0} \dot{w}^{\text {int }}=\rho_{0} \frac{\partial w^{\text {int }}(\mathbf{X}, t)}{\partial t}=\dot{\mathbf{F}}^{T}: \mathbf{P}-\nabla_{X} \cdot \tilde{\mathbf{q}}+\rho_{0} s \tag{B3.3.9}
\end{equation*}
$$

3.5.11 System Equations. The number of dependent variables depends on the number of space dimensions in the model. If we denote the number of space dimensions by $n_{S D}$, then for a purely mechanical problem, the following unknowns occur in the equations for a purely mechanical process (a process without heat transfer, so the energy equation is not used):
$\rho$, the density 1 unknown
$\mathbf{v}$, the velocity $\quad n_{S D}$ unknowns
$\sigma$, the stresses

$$
n_{\sigma}=n_{S D} *\left(n_{S D^{+}}+1\right) / 2 \text { unknowns }
$$

In counting the number of unknowns attributed to the stress tensor, we have exploited its symmetry, which results from the conservation of angular momentum. The combination of the mass conservation ( 1 equation), and the momentum conservation ( $n_{S D}$ equations) gives a total of $n_{S D}+1$ equations. Thus we are left with $n_{\sigma}$ extra unknowns. These are provided by the constitutive equations, which relate the stresses to a measure of deformation. This equation introduces $n_{\sigma}$ additional unknowns, the components of the symmetric rate-ofdeformation tensor. However, these unknowns can immediately be expressed in terms of the velocities by Eq. (3.3.10), so they need not be counted as additional unknowns.

The displacements are not counted as unknowns. The displacements are considered secondary dependent variables since they can be obtained by integrating the velocities in time using Eq. (3.2.8) at any material point. The displacements are considered secondary dependent variables, just like the position vectors. This choice of dependent variables is a matter of preference. We could just as easily have chosen the displacement as a primary dependent variable and the velocity as a secondary dependent variable.

### 3.6. LAGRANGIAN CONSERVATION EQUATIONS

3.6.1 Introduction and Definitions. For solid mechanics applications, it is instructive to directly develop the conservation equations in terms of the Lagrangian measures of stress and strain in the reference configuration. In the continuum mechanics literature such formulations are called Lagrangian, whereas in the finite element literature these formulations are called total Lagrangian formulations. For a total Lagrangian formulation, a Lagrangian mesh is always used. The conservation equations in a Lagrangian framework are fundamentally identical to those which have just been developed, they are just expressed in terms
of different variables. In fact, as we shall show, they can be obtained by the transformations in Box 3.2 and the chain rule. This Section can be skipped in a first reading. It is included here because much of the finite element literature for nonlinear mechanics employs total Lagrangian formulations, so it is essential for a serious student of the field.

The independent variables in the total Lagrangian formulation are the Lagrangian (material) coordinates $\mathbf{X}$ and the time $t$. The major dependent variables are the initial density $\rho_{0}(\mathbf{X}, t)$ the displacement $\mathbf{u}(\mathbf{X}, t)$ and the Lagrangian measures of stress and strain. We will use the nominal stress $\mathbf{P}(\mathbf{X}, t)$ as the measure of stress. This leads to a momentum equation which is strikingly similar to the momentum equation in the Eulerian description, Eq. (3.5.33), so it is easy to remember. The deformation will be described by the deformation gradient $\mathbf{F}(\mathbf{X}, t)$. The pair $\mathbf{P}$ and $\mathbf{F}$ is not especially useful for constructing constitutive equations, since $\mathbf{F}$ does not vanish in rigid body motion and $\mathbf{P}$ is not symmetric. Therefore constitutive equations are usually formulated in terms of the of the PK2 stress $\mathbf{S}$ and the Green strain $\mathbf{E}$. However, keep in mind that relations between $\mathbf{S}$ and $\mathbf{E}$ can easily be transformed to relations between $\mathbf{P}$ and $\mathbf{E}$ or $\mathbf{F}$ by use of the relations in Boxes 3.2.

The applied loads are defined on the reference configuration. The traction $\mathbf{t}_{0}$ is defined in Eq. (3.4.2); $\mathbf{t}_{0}$ is in units of force per unit initial area. As mentioned in Chapter 1, we place the noughts, which indicate that the variables pertain to the reference configuration, either as subscripts or superscripts, whichever is convenient. The body force is denoted by $\mathbf{b}$, which is in units of force per unit mass; the body force per initial unit volume is given by $\rho_{0} \mathbf{b}$, which is equivalent to the force per unit current volume $\rho \mathbf{b}$. This equivalence is shown in the following

$$
\begin{equation*}
d \mathbf{f}=\rho \mathbf{b} d \Omega=\rho \mathbf{b} J d \Omega_{0}=\rho_{0} \mathbf{b} d \Omega_{0} \tag{3.6.1}
\end{equation*}
$$

where the second equality follows from the conservation of mass, Eq. (3.5.25). Many authors, including Malvern(1969) use different symbols for the body forces in the two formulations; but this is not necessary with our convention of associating symbols with fields.

The conservation of mass has already been developed in a form that applies to the total Lagrangian formulation, Eq.(3.5.25). Therefore we develop only the conservation of momentum and energy.
3.6.2 Conservation of Linear Momentum. In a Lagrangian description, the linear momentum of a body is given in terms of an integral over the reference configuration by

$$
\begin{equation*}
\mathbf{p}_{0}(t)=\int_{\Omega_{0}} \rho_{0} \mathbf{v}(\mathbf{X}, t) d \Omega_{0} \tag{3.6.2}
\end{equation*}
$$

The total force on the body is given by integrating the body forces over the reference domain and the traction over the reference boundaries:

$$
\begin{equation*}
\mathbf{f}_{0}(t)=\int_{\Omega_{0}} \rho_{0} \mathbf{b}(\mathbf{X}, t) d \Omega_{0}+\int_{\Gamma_{0}} \mathbf{t}_{0}(\mathbf{X}, t) d \Gamma_{0} \tag{3.6.3}
\end{equation*}
$$

Newton's second law then gives

$$
\begin{equation*}
\frac{d \mathbf{p}_{0}}{d t}=\mathbf{f}_{0} \tag{3.6.4}
\end{equation*}
$$

Substituting (3.6.2) and (3.6.3) into the above gives

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega_{0}} \rho_{0} \mathbf{v} d \Omega_{0}=\int_{\Omega_{0}} \rho_{0} \mathbf{b} d \Omega_{0}+\int_{\Gamma_{0}} \mathbf{t}_{0} d \Gamma_{0} \tag{3.6.5}
\end{equation*}
$$

On the LHS, the material derivative can be taken inside the integral because the reference domain is constant in time, so

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega_{0}} \rho_{0} \mathbf{v} d \Omega_{0}=\int_{\Omega_{0}} \rho_{0} \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t} d \Omega_{0} \tag{3.6.6}
\end{equation*}
$$

Using Cauchy's law (3.4.2) and Gauss' theorem in sequence gives

$$
\begin{align*}
& \int_{\Gamma_{0}} \mathbf{t}_{0} d \Gamma_{0}=\int_{\Gamma_{0}} \mathbf{n}_{0} \cdot \mathbf{P} d \Gamma_{0}=\int_{\Omega_{0}} \nabla_{\mathbf{X}} \cdot \mathbf{P} d \Omega_{0} \quad \text { or } \\
& \int_{\Gamma_{0}} t_{i}^{0} d \Gamma_{0}=\int_{\Gamma_{0}} n_{j}^{0} P_{j i} d \Gamma_{0}=\int_{\Omega_{0}} \frac{\partial P_{j i}}{\partial X_{j}} d \Omega_{0} \tag{3.6.7}
\end{align*}
$$

Note that in tensor notation, the left gradient appears in the domain integral because the nominal stress is defined with the normal on the left side. The definition of the material gradient, which is distinguished with the subscript $\mathbf{X}$, should be clear from the indicial expression. The index on the material coordinate is the same as the first index on the nominal stress: the order is important because the nominal stress is not symmetric.

Substituting (3.6.6) and (3.6.7) into (3.6.5) gives

$$
\begin{equation*}
\int_{\Omega_{0}}\left(\rho_{0} \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}-\rho_{0} \mathbf{b}-\nabla_{\mathbf{X}} \cdot \mathbf{P}\right) d \Omega_{0}=0 \tag{3.6.8}
\end{equation*}
$$

which, because of the arbitrariness of $\Omega_{0}$ gives

$$
\begin{equation*}
\rho_{0} \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}=\nabla_{\mathbf{X}} \cdot \mathbf{P}+\rho_{0} \mathbf{b} \quad \text { or } \quad \rho_{0} \frac{\partial v_{i}(\mathbf{X}, t)}{\partial t}=\frac{\partial P_{j i}}{\partial X_{j}}+\rho_{0} b_{i} \tag{3.6.9}
\end{equation*}
$$

Comparing the above with the momentum equation in the Eulerian description, Eq.(3.5.33), we can see that they are quite similar: in the Lagrangian form of the
momentum equation the Cauchy stress is replaced by the nominal stress and the density is replaced by the density in the reference configuration.

The above form of the momentum equation can also be obtained directly by transforming all of the terms in Eq.(3.5.33) using the chain rule and Box 3.2. Actually, this is somewhat difficult, particularly for the gradient term. Using the transformation from Box 3.2 and the chain rule gives

$$
\begin{equation*}
\frac{\partial \sigma_{j i}}{\partial x_{j}}=\frac{\partial\left(J^{-1} F_{j k} P_{k i}\right)}{\partial x_{j}}=P_{k i} \frac{\partial}{\partial x_{j}}\left(J^{-1} F_{j k}\right)+J^{-1} F_{j k} \frac{\partial P_{k i}}{\partial x_{j}}=J^{-1} \frac{\partial x_{j}}{\partial x_{k}} \frac{\partial P_{k i}}{\partial x_{j}} \tag{3.6.10}
\end{equation*}
$$

In the above we have used the definition of the deformation gradient $\mathbf{F}$, Eq. (3.2.14) and $\partial\left(J^{-1} F_{j k}\right) / \partial x_{j}=0$,(see Ogden(1984)). Thus (3.5.33) becomes

$$
\begin{equation*}
\rho \frac{\partial v_{i}}{\partial t}=J^{-1} \frac{\partial x_{j}}{\partial X_{k}} \frac{\partial P_{k i}}{\partial x_{j}}+\rho b_{i} \tag{3.6.11}
\end{equation*}
$$

By the chain rule, the first term on the RHS is $J^{-1} \partial P_{k i} / \partial X_{k}$. Multiplying the equation by $J$ and using mass conservation, $\rho J=\rho_{0}$ then gives Eq. (3.6.9).
3.6.3 Conservation of Angular Momentum. The balance equations for angular momentum will not be rederived in the total Lagrangian framework. We will use the consequence of angular momentum balance in Eq. (3.5.40) in conjunction with the stress transformations in Box 3.2 to derive the consequences for the Lagrangian measures of stress. Substituting the transformation expression from Box 3.2 into (3.5.40) gives

$$
\begin{equation*}
J^{-1} \mathbf{F} \cdot \mathbf{P}=\left(J^{-1} \mathbf{F} \cdot \mathbf{P}\right)^{T} \tag{3.6.12}
\end{equation*}
$$

Multiplying both sides of the above by $J$ and taking the transpose inside the parenthesis then gives

$$
\begin{equation*}
\mathbf{F} \cdot \mathbf{P}=\mathbf{P}^{T} \cdot \mathbf{F}^{T} \quad F_{i k} P_{k j}=P_{i k}^{T} F_{k j}^{T}=F_{j k} P_{k i} \tag{3.6.13}
\end{equation*}
$$

The above equations are nontrivial only when $i \neq j$. Thus the above gives one nontrivial equation in two dimensions, three nontrivial equations in three dimensions. So, while the nominal stress is not symmetric, the number of conditions imposed by angular momentum balance equals the number of symmetry conditions on the Cauchy stress, Eq. (3.5.40). In two dimensions, the angular momentum equation is

$$
\begin{equation*}
F_{11} P_{12}+F_{12} P_{22}=F_{21} P_{11}+F_{22} P_{21} \tag{3.6.14}
\end{equation*}
$$

These conditions are usually imposed directly on the constitutive equation, as will be seen in Chapter 5.

For the PK2 stress, the conditions emanating from conservation of angular momentum can be obtained by expressing $\mathbf{P}$ in terms of $\mathbf{S}$ in Eq. (3.6.13), (the same equations are obtained if $\sigma$ is replaced by $\mathbf{S}$ in the symmetry conditions (3.5.40)), which gives

$$
\begin{equation*}
\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T}=\mathbf{F} \cdot \mathbf{S}^{T} \cdot \mathbf{F}^{T} \tag{3.6.15}
\end{equation*}
$$

Since $\mathbf{F}$ must be a regular (nonsingular) matrix, its inverse exists and we can premultiply by $\mathbf{F}^{-1}$ and postmultiply by $\mathbf{F}^{-T} \equiv\left(\mathbf{F}^{-1}\right)^{T}$ the above to obtain

$$
\begin{equation*}
\mathbf{S}=\mathbf{S}^{T} \tag{3.6.16}
\end{equation*}
$$

So the conservation of angular momentum requires the PK2 stress to be symmetric.
3.6.4 Conservation of Energy in Lagrangian Description. The counterpart of Eq. (3.5.45) in the reference configuration can be written as

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega_{0}}\left(\rho_{0} w^{\text {int }}+\frac{1}{2} \rho_{0} \mathbf{v} \cdot \mathbf{v}\right) d \Omega_{0}= \\
& \quad \int_{\Omega_{0}} \mathbf{v} \cdot \rho_{0} \mathbf{b} d \Omega_{0}+\int_{\Gamma_{0}} \mathbf{v} \cdot \mathbf{t}_{0} d \Gamma_{0}+\int_{\Omega_{0}} \rho_{0} s d \Omega_{0}-\int_{\Gamma_{0}} \mathbf{n}_{0} \cdot \tilde{\mathbf{q}} d \Gamma_{0} \tag{3.6.17}
\end{align*}
$$

The heat flux in a total Lagrangian formulation is defined as energy per unit reference area and therefore is denoted by $\tilde{\mathbf{q}}$ to distinguish it from the heat flux per unit current area $\mathbf{q}$, which are related by

$$
\begin{equation*}
\tilde{\mathbf{q}}=J^{-1} \mathbf{F}^{T} \cdot \mathbf{q} \tag{3.6.17b}
\end{equation*}
$$

The above follows from Nanson's law (3.4.5) and the equivalence

$$
\int_{\Gamma} \mathbf{n} \cdot \mathbf{q} d \Gamma=\int_{\Gamma_{0}} \mathbf{n}_{0} \cdot \tilde{\mathbf{q}} d \Gamma_{0}
$$

Substituting (3.4.5) for $\mathbf{n}$ into the above gives (3.6.17b).
The internal energy per unit initial volume in the above is related to the internal energy per unit current volume in (3.5.45) as follows

$$
\begin{equation*}
\rho_{0} w^{\text {int }} d \Omega_{0}=\rho_{0} w^{\text {int }} J^{-1} d \Omega=\rho w^{\text {int }} d \Omega \tag{3.6.18}
\end{equation*}
$$

where the last step follows from the mass conservation equation (3.5.9). On the LHS, the time derivative can be taken inside the integral since the domain is fixed, giving

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega_{0}}\left(\rho_{0} w^{i n t}+\frac{1}{2} \rho_{0} \mathbf{v} \cdot \mathbf{v}\right) d \Omega_{0}=\int_{\Omega_{0}}\left(\rho_{0} \frac{\partial w^{i n t}(\mathbf{X}, t)}{\partial t}+\rho_{0} \mathbf{v} \cdot \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}\right) d \Omega_{0}( \tag{3.6.19}
\end{equation*}
$$

The second term on the RHS can be modified as follows by using Eq. (3.4.2) and Gauss's theorem

$$
\begin{align*}
\int_{\Gamma_{0}} \mathbf{v} \cdot \mathbf{t}_{0} d \Gamma_{0} & =\int_{\Gamma_{0}} v_{j} t_{j}^{0} d \Gamma_{0}=\int_{\Gamma_{0}} v_{j} n_{i}^{0} P_{i j} d \Gamma_{0} \\
& =\int_{\Omega_{0}} \frac{\partial}{\partial X_{i}}\left(v_{j} P_{i j}\right) d \Omega_{0}=\int_{\Omega_{0}}\left(\frac{\partial v_{j}}{\partial X_{i}} P_{i j}+v_{j} \frac{\partial P_{i j}}{\partial X_{i}}\right) d \Omega_{0}  \tag{3.6.20}\\
& =\int_{\Omega_{0}}\left(\frac{\partial F_{j i}}{\partial t} P_{i j}+\frac{\partial P_{i j}}{\partial X_{i}} v_{j}\right) d \Omega_{0}=\int_{\Omega_{0}}\left(\frac{\partial \mathbf{F}^{T}}{\partial t}: \mathbf{P}+\left(\nabla_{X} \cdot \mathbf{P}\right) \cdot \mathbf{v}\right) d \Omega_{0}
\end{align*}
$$

Gauss's theorem on the fourth term of the LHS and some manipulation gives

$$
\begin{equation*}
\int_{\Omega_{0}}\left(\rho_{0} \frac{\partial w^{i n t}}{\partial t}-\frac{\partial \mathbf{F}^{T}}{\partial t}: \mathbf{P}+\nabla_{X} \cdot \tilde{\mathbf{q}}-\rho_{0} s+\left(\rho_{0} \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}-\nabla_{X} \cdot \mathbf{P}-\rho_{0} \mathbf{b}\right) \cdot \mathbf{v}\right) d \Omega_{0}=0 \tag{3.6.21}
\end{equation*}
$$

The term inside the parenthesis of the integrand is the total Lagrangian form of the momentum equation, (3.6.30), so it vanishes. Then because of the arbitrariness of the domain, the integrand vanishes, giving

$$
\begin{equation*}
\rho_{0} \dot{w}^{i n t}=\rho_{0} \frac{\partial w^{i n t}(\mathbf{X}, t)}{\partial t}=\dot{\mathbf{F}}^{T}: \mathbf{P}-\nabla_{X} \cdot \tilde{\mathbf{q}}+\rho_{0} s \tag{3.6.22}
\end{equation*}
$$

In the absence of heat conduction or heat sources, the above gives

$$
\begin{equation*}
\rho_{0} \dot{w}^{i n t}=\dot{F}_{j i} P_{i j}=\dot{\mathbf{F}}^{T}: \mathbf{P}=\mathbf{P}: \dot{\mathbf{F}} \tag{3.6.23}
\end{equation*}
$$

This is the Lagrangian counterpart of Eq. (3.5.50). It shows that the nominal stress is conjugate in power to the material time derivative of the deformation gradient.

These energy conservation equations could also be obtained directly from Eq. (3.5.50) by transformations. This is most easily done in indicial notation.

$$
\begin{aligned}
\rho D_{i j} \sigma_{i j} & =\rho \frac{\partial v_{i}}{\partial x_{j}} \sigma_{i j} \quad \text { by definition of } \mathbf{D} \text { and symmetry of stress } \sigma \\
& =\rho \frac{\partial v_{i}}{\partial X_{k}} \frac{\partial X_{k}}{\partial x_{j}} \sigma_{i j} \quad \text { by chain rule }
\end{aligned}
$$

$$
\begin{align*}
& =\rho \dot{F}_{i k} \frac{\partial X_{k}}{\partial x_{j}} \sigma_{i j} \quad \text { by definition of } \mathbf{F}, \text { Eq. (3.2.10) }  \tag{3.6.24}\\
& =\rho \dot{F}_{i k} P_{k i} J^{-1}=\rho_{0} \dot{F}_{i k} P_{k i} \quad \text { by Box } 3.2 \text { and mass conservation }
\end{align*}
$$

3.6.5 Power in terms of PK2 stress. The stress transformations in Box 3.2 can also be used to express the internal energy in terms of the PK2 stress.

$$
\begin{aligned}
\dot{\mathbf{F}}^{T}: \mathbf{P} \equiv & \dot{F}_{i k} P_{k i}=\dot{F}_{i k} S_{k r} F_{r i}^{T} \quad \text { by Box } 3.2 \\
= & F_{r i}^{T} \dot{F}_{i k} S_{r k}=\left(\mathbf{F}^{\mathbf{T}} \cdot \dot{\mathbf{F}}\right): \mathbf{S} \quad \text { by symmetry of } \mathbf{S} \\
= & \left(1 / 2\left(\mathbf{F}^{\mathbf{T}} \cdot \dot{\mathbf{F}}+\dot{\mathbf{F}}^{\mathbf{T}} \cdot \mathbf{F}\right)+1 / 2\left(\mathbf{F}^{\mathbf{T}} \cdot \dot{\mathbf{F}}-\dot{\mathbf{F}}^{\mathbf{T}} \cdot \mathbf{F}\right)\right): \mathbf{S} \quad \text { decomposing } \\
& \quad \text { tensor into symmetric and antisymmetric parts } \\
= & 1 / 2\left(\mathbf{F}^{\mathbf{T}} \dot{\mathbf{F}}+\dot{\mathbf{F}}^{\mathbf{T}} \mathbf{F}\right): \mathbf{S} \quad \text { since contraction of symmetric and } \\
& \text { antisymmetric tensors vanishes }
\end{aligned}
$$

Then, using the time derivative of $\mathbf{E}$ as defined in Eq.(3.3.20) gives

$$
\begin{equation*}
\rho_{0} \dot{w}^{i n t}=\dot{\mathbf{E}}: \mathbf{S}=\mathbf{S}: \dot{\mathbf{E}}=\dot{E}_{i j} S_{i j} \tag{3.6.26}
\end{equation*}
$$

This shows that the rate of the Green strain tensor is conjugate in power (or energy) to the PK2 stress.

Thus we have identified three stress and strain rate measures which are conjugate in the sense of power. These conjugate measures are listed in Box 3.4 along with the corresponding expressions for the power. Box 3.4 also includes a fourth conjugate pair, the corotational Cauchy stress and corotational rate-ofdeformation. Its equivalence to the power in terms of the unrotated Cauchy stress and rate-of-deformation is easily demonstrated by (3.4.15) and thhe orthgonality of the rotation matrix.

Conjugate stress and strain rate measures are useful in developing weak forms of the momentum equation, i.e. the principles of virtual work and power. The conjugate pairs presented here just scratch the surface: many other conjugate pairs have been developed in continuum mechanics, \{Ogden(1984), Hill()\}. However, those presented here are the most frequently used in nonlinear finite element methods.

## Box 3.4

## Stress-deformation (strain) rate pairs conjugate in power

Cauchy stress/rate-of deformation: $\rho \dot{w^{i n t}}=\mathbf{D}: \sigma=\sigma: \mathbf{D}=D_{i j} \sigma_{i j}$

Nominal stress/rate of deformation gradient: $\rho_{0} \dot{w}^{\text {int }}=\dot{\mathbf{F}}: \mathbf{P}^{T}=\mathbf{P}: \dot{\mathbf{F}}^{T}=\dot{F}_{i j} P_{j i}$
PK2 stress/rate of Green strain: $\rho_{0} \dot{w}^{\text {int }}=\dot{\mathbf{E}}: \mathbf{S}=\mathbf{S}: \dot{\mathbf{E}}=\dot{E}_{i j} S_{i j}$
Corotational Cauchy stress/rate-of-deformation: $\rho \dot{w}{ }^{i n t}=\hat{\mathbf{D}}: \hat{\sigma}=\hat{\sigma}: \hat{\mathbf{D}}=\hat{D}_{i j} \hat{\sigma}_{i j}$

### 3.7 POLAR DECOMPOSITION AND FRAME-INVARIANCE

In this Section, the role of rigid body rotation is explored. First, a theorem known as the polar decomposition theorem is presented. This theorem enables the rigid body rotation to be obtained for any deformation. Next, we consider the effect of rigid body rotations on constitutive equations. We show that for the Cauchy stress, a modification of the time derivatives is needed to formulate rate constitutive equations. This is known as a frame-invariant or objective rate of stress. Three frame-invariant rates are presented: the Jaumann rate, the Truesdell rate and the Green-Naghdi rate. Some startling differences in hypoelastic constitutive equations with these various rates are then demonstrated.
3.7.1 Polar Decomposition Theorem. A fundamental theorem which elucidates the role of rotation in large deformation problems is the polar decomposition theorem. In continuum mechanics, this theorem states that any deformation gradient tensor $\mathbf{F}$ can be multiplicatively decomposed into the product of an orthogonal matrix $\mathbf{R}$ and a symmetric tensor $\mathbf{U}$, called the right stretch tensor (the adjective right is often omitted):

$$
\begin{array}{lll}
\mathbf{F}=\mathbf{R} \cdot \mathbf{U} & \text { or } & F_{i j}=\frac{\partial x_{i}}{\partial X_{j}}=R_{i k} U_{k j} \quad \text { where } \\
\mathbf{R}^{-1}=\mathbf{R}^{T} & \text { and } & \mathbf{U}=\mathbf{U}^{T} \tag{3.7.2}
\end{array}
$$

Rewriting the above with Eq. (3.2.15) gives

$$
\begin{equation*}
d \mathbf{x}=\mathbf{R} \cdot \mathbf{U} \cdot d \mathbf{X} \tag{3.7.3}
\end{equation*}
$$

The above shows that any motion of a body consists of a deformation, which is represented by the symmetric mapping $\mathbf{U}$, and a rigid body rotation $\mathbf{R} ; \mathbf{R}$ can be recognized as a rigid-body rotation because all proper orthogonal transformations are rotations. Rigid body translation does not appear in this equation because $d \mathbf{x}$ and $d \mathbf{X}$ are differential line segments in the current and reference configurations, respectively, and the differential line segments are not affected by translation. If Eq. (3.7.3) were integrated to obtain the deformation function, $\mathbf{x}=\phi(\mathbf{X}, t)$, then the rigid body translation would appear as a constant of integration. In a translation, $\mathbf{F}=\mathbf{I}$, and $d \mathbf{x}=d \mathbf{X}$.

The polar decomposition theorem is proven in the following. To simplify the proof, we treat the tensors as matrices. Premultiplying both sides of Eq. (3.7.1) by its transpose gives

$$
\begin{equation*}
\mathbf{F}^{T} \cdot \mathbf{F}=(\mathbf{R U})^{T}(\mathbf{R} \mathbf{U})=\mathbf{U}^{T} \mathbf{R}^{T} \mathbf{R} \mathbf{U}=\mathbf{U}^{T} \mathbf{U}=\mathbf{U} \mathbf{U} \tag{3.7.4}
\end{equation*}
$$

where (3.7.2) is used to obtain the third and fourth equalities. The last term is the square of the $\mathbf{U}$ matrix. It follows that

$$
\begin{equation*}
\mathbf{U}=\left(\mathbf{F}^{T} \cdot \mathbf{F}\right)^{1 / 2} \tag{3.7.5}
\end{equation*}
$$

The fractional power of a matrix is defined in terms of its spectral representation, see e.g. Chandrasekharaiah and Debnath (1994, p96). It is computed by first transforming the matrix to its principal coordinates, where the matrix becomes a diagonal matrix with the eigenvalues on the diagonal. The fractional power is then applied to all of the diagonal terms, and the matrix is transformed back. This is illustrated in the following examples. The matrix $\mathbf{F}^{T} \cdot \mathbf{F}$ is positive definite, so all of its eigenvalues are positive. Consequently the matrix $\mathbf{U}$ is always real.

The rotation part of the deformation, $\mathbf{R}$, can then be found by applying Eq. (3.7.1), which gives

$$
\begin{equation*}
\mathbf{R}=\mathbf{F} \cdot \mathbf{U}^{-1} \tag{3.7.6}
\end{equation*}
$$

The existence of the inverse of $\mathbf{U}$ follows from the fact that all of its eigenvalues are always positive, since the right hand side of Eq. (3.7.5) is always a positive matrix.

The matrix $\mathbf{U}$ is closely related to an engineering definition of strain. Its principal values represent the elongations of line segments in the principal directions of $\mathbf{U}$. Therefore, many researchers have found this tensor to be appealing for developing constitutive equations. The tensor $\mathbf{U}-\mathbf{I}$ is called the Biot strain tensor.

A deformation can also be decomposed in terms of a left stretch tensor and a rotation according to

$$
\begin{equation*}
\mathbf{F}=\mathbf{V} \cdot \mathbf{R} \tag{3.7.7}
\end{equation*}
$$

This form of the polar decomposition is used less frequently and we only note it in passing here. It will play a role in discussions of material symmetry for elastic materials at finite strain. The polar decomposition theorem, which is usually applied to the deformation tensor, applies to any invertible square matrix: any square matrix can be multiplicatively decomposed into a rotation matrix and a symmetric matrix, see Chandrasekharaiah and Debnath (1994, p97).

It is emphasized that the rotations of different line segments at the same point depend on the orientation of the line segment. In a three dimensional body, only three line segments are rotated exactly by $\mathbf{R}(\mathbf{X}, t)$ at any point $\mathbf{X}$. These are the line segments corresponding to the principal directions of the stretch tensor $\mathbf{U}$. It can be shown that these are also the principal directions of the Green strain tensor. The rotations of line segments which are oriented in directions other than the principal directions of $\mathbf{E}$ are not given by $\mathbf{R}$.

Example 3.10 Consider the motion of a triangular element in which the nodal coordinates $x_{I}(t)$ and $y_{I}(t)$ are given by

$$
\begin{array}{ll}
x_{1}(t)=a+2 a t & y_{1}(t)=2 a t \\
x_{2}(t)=2 a t & y_{2}(t)=2 a-2 a t  \tag{E3.10.1}\\
x_{3}(t)=3 a t & y_{3}(t)=0
\end{array}
$$

Find the rigid body rotation and the stretch tensors by the polar decomposition theorem at $t=1.0$ and at $t=0.5$.

The motion of a triangular domain can most easily be expressed by using the shape functions for triangular elements, i.e. the area coordinates. In terms of the triangular coordinates, the motion is given by

$$
\begin{align*}
& x(\xi, t)=x_{1}(t) \xi_{1}+x_{2}(t) \xi_{2}+x_{3}(t) \xi_{3}  \tag{E3.10.2}\\
& y(\xi, t)=y_{1}(t) \xi_{1}+y_{2}(t) \xi_{2}+y_{3}(t) \xi_{3} \tag{E3.10.3}
\end{align*}
$$

where $\xi_{I}$ are the triangular, or area, coordinates; see Appendix A; the material coordinates appear implicitly in the RHS of the above through the relationship between the area coordinates and the coordinates at time $t=0$. To extract those relationships we write the above at time $t=0$, which gives

$$
\begin{align*}
& x(\xi, 0)=X=X_{1} \xi_{1}+X_{2} \xi_{2}+X_{3} \xi_{3}=a \xi_{1}  \tag{E3.10.4}\\
& y(\xi, 0)=Y=Y_{1} \xi_{1}+Y_{2} \xi_{2}+Y_{3} \xi_{3}=2 a \xi_{2} \tag{E3.10.5}
\end{align*}
$$

In this case, the relations between the triangular coordinates are particularly simple because most of the nodal coordinates vanish in the initial configuration, so the relations developed above could be obtained by inspection.

Using Eq. (E3.10.5) to express the triangular coordinates in terms of the material coordinates, Eq (E3.10.1) can be written

$$
\begin{align*}
x(\mathbf{X}, 1) & =3 a \xi_{1}+2 a \xi_{2}+3 a \xi_{3} \\
& =3 X+Y+3 a\left(1-\frac{X}{a}-\frac{Y}{2 a}\right)=3 a-\frac{Y}{2}  \tag{E3.10.6}\\
y(\mathbf{X}, 1) & =2 a \xi_{1}+0 \xi_{2}+0 \xi_{3}  \tag{E3.10.7}\\
& =2 X
\end{align*}
$$

The deformation gradient is then obtained by evaluating the derivatives of the above motion using Eq. (3.2.16)

$$
\mathbf{F}=\left[\begin{array}{ll}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y}  \tag{E3.10.8}\\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y}
\end{array}\right]=\left[\begin{array}{cc}
0 & -0.5 \\
2 & 0
\end{array}\right]
$$

The stretch tensor $\mathbf{U}$ is then evaluated by Eq. (3.7.5):

$$
\mathbf{U}=\left(\mathbf{F}^{T} \mathbf{F}\right)^{1 / 2}=\left[\begin{array}{cc}
4 & 0  \tag{E3.10.9}\\
0 & 0.25
\end{array}\right]^{1 / 2}=\left[\begin{array}{cc}
2 & 0 \\
0 & 0.5
\end{array}\right]
$$

In this case the $\mathbf{U}$ matrix is diagonal, so the principal values are simply the diagonal terms. The positive square roots are chosen in evaluating the square root of the matrix because the principal stretches must be positive; otherwise the Jacobian determinant would be negative since according to Eq. (3.7.1), $J=\operatorname{det}(\mathbf{R}) \operatorname{det}(\mathbf{U})$ and $\operatorname{det}(\mathbf{R})=1$, so $\operatorname{det}(\mathbf{U})<0$ implies $J<0$. The rotation matrix $\mathbf{R}$ is then given by Eq. (3.7.6):

$$
\mathbf{R}=\mathbf{F U}^{-1}=\left[\begin{array}{cc}
0 & -0.5  \tag{E3.10.10}\\
2 & 0
\end{array}\right]\left[\begin{array}{cc}
0.5 & 0 \\
0 & 2
\end{array}\right]=\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]
$$

Comparing the above rotation matrix $\mathbf{R}$ and Eq. (3.2.25), it can be seen that the rotation is a counterclockwise 90 degree rotation. This is also readily apparent from Fig. 3.9. The deformation consists of an elongation of the line segment between nodes 1 and 3, i.e. $d X$, by a factor of 2, (see $\mathbf{U}_{11}$ in Eq. (E3.10.9)) and a contraction of the line segment between nodes 3 and 2, i.e. $d Y$, by a factor of 0.5 , (see $\mathbf{U}_{22}$ in Eq. (E3.10.9)), followed by a translation of $3 a$ in the x-direction and a 90 degree rotation. Since the original line segments along the $x$ and $y$ directions correspond to the principal directions, or eigenvectors, of $\mathbf{U}$, the rotations of these line segments correspond to the rotation of the body in the polar decomposition theorem.

The configuration at $t=0.5$ is given by evaluating Eq. (E3.10.1) at that time, giving:

$$
\begin{align*}
x(\mathbf{X}, 0.5) & =2 a \xi_{1}+a \xi_{2}+1.5 a \xi_{3} \\
= & 2 a \frac{X}{a}+a \frac{Y}{2 a}+1.5 a\left(1-\frac{X}{a}-\frac{Y}{2 a}\right)=1.5 a+0.5 X-0.25 Y  \tag{E3.10.11a}\\
y(\mathbf{X}, 0.5) & =a \xi_{1}+a \xi_{2}+0 \xi_{3} \\
= & a \frac{X}{a}+a \frac{Y}{2 a}=X+0.5 Y \tag{E3.10.11b}
\end{align*}
$$

The deformation gradient $\mathbf{F}$ is then given by

$$
\mathbf{F}=\left[\begin{array}{ll}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y}  \tag{E3.10.12}\\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y}
\end{array}\right]=\left[\begin{array}{cc}
0.5 & -0.25 \\
1 & 0.5
\end{array}\right]
$$

and the stretch tensor $\mathbf{U}$ is given by Eq. (3.7.6):

$$
\mathbf{U}=\left(\mathbf{F}^{T} \mathbf{F}\right)^{1 / 2}=\left[\begin{array}{cc}
1.25 & 0.375  \tag{E3.10.13}\\
0.375 & 0.3125
\end{array}\right]^{1 / 2}=\left[\begin{array}{ll}
1.0932 & 0.2343 \\
0.2343 & 0.5076
\end{array}\right]
$$

The last matrix in the above is obtained by finding the eigenvalues $\lambda_{i}$ of $\mathbf{F}^{T} \mathbf{F}$, taking their positive square roots, and placing them on a diagonal matrix called $\mathbf{H}=\operatorname{diag}\left(\sqrt{\lambda_{1}}, \sqrt{\lambda_{2}}\right)$. The matrix $\mathbf{H}$ transformed back to the global components by $\mathbf{U}=\mathbf{A}^{T} \mathbf{H A}$ where $\mathbf{A}$ is the matrix whose columns are the eigenvectors of $\mathbf{F}^{T} \mathbf{F}$. These matrices are:

$$
\mathbf{A}=\left\lfloor\begin{array}{cc}
-0.9436 & 0.3310  \tag{E3.10.14}\\
-03310 & -0.9436
\end{array}\right\rfloor \quad \mathbf{H}=\left\lfloor\begin{array}{cc}
13815 & 0 \\
0 & 0.1810
\end{array}\right\rfloor
$$

The rotation matrix $\mathbf{R}$ is then found by

$$
\mathbf{R}=\mathbf{F U}^{-1}=\left[\begin{array}{cc}
0.5 & -0.25  \tag{E3.10.15}\\
1 & 0.5
\end{array}\right]\left[\begin{array}{ll}
1.0932 & 0.2343 \\
0.2343 & 0.5076
\end{array}\right]^{-1}=\left[\begin{array}{cc}
0.6247 & -0.7809 \\
0.7809 & 0.6247
\end{array}\right]
$$

Example 3.11 Consider the deformation for which the deformation gradient is

$$
\begin{align*}
& \mathbf{F}=\left[\begin{array}{ll}
c-a s & a c-s \\
s+a c & a s+c
\end{array}\right]  \tag{E3.11.1}\\
& c=\cos \theta, \\
& s=\sin \theta
\end{align*}
$$

where $a$ is a constant. Find the stretch tensor and the rotation matrix when $a=\frac{1}{2}$, $\theta=\frac{\pi}{2}$.

For the particular values given

$$
\mathbf{F}=\left[\begin{array}{cc}
-\frac{1}{2} & -1  \tag{E3.11.2}\\
1 & \frac{1}{2}
\end{array}\right] \quad \mathbf{C}=\mathbf{F}^{T} \cdot \mathbf{F}=\left[\begin{array}{cc}
1.25 & 1 \\
1 & 125
\end{array}\right]
$$

The eigenvalues and corresponding eigenvectors of $\mathbf{C}$ are

$$
\begin{array}{ll}
\lambda_{1}=025 & \mathbf{y}_{1}^{T}=\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & -1
\end{array}\right] \\
\lambda_{2}=225 & \mathbf{y}_{2}^{T}=\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & 1
\end{array}\right] \tag{E3.11.3}
\end{array}
$$

The diagonal form of $\mathbf{C}, \operatorname{diag}(\mathbf{C})$, consists of these eigenvalues and the square root of $\operatorname{diag}(\mathbf{C})$ is obtained by taking the positive square roots of these eigenvalues

$$
\operatorname{diag}(\mathbf{C})=\left[\begin{array}{ll}
\frac{1}{4} & 0  \tag{E3.11.4}\\
0 & \frac{9}{4}
\end{array}\right] \Rightarrow \operatorname{diag}\left(\mathbf{C}^{1 / 2}\right)=\left[\begin{array}{ll}
\frac{1}{2} & 0 \\
0 & \frac{3}{2}
\end{array}\right]
$$

The $\mathbf{U}$ matrix is then obtained by transforming $\operatorname{diag}(\mathbf{C})$ back to the $x-y$ coordinate system

$$
\mathbf{U}=\mathbf{Y} \cdot \operatorname{diag}\left(\mathbf{C}^{1 / 2}\right) \cdot \mathbf{Y}^{T}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1  \tag{E3.11.5}\\
-1 & 1
\end{array}\right]\left[\begin{array}{ll}
\frac{1}{2} & 0 \\
0 & \frac{3}{2}
\end{array}\right]^{\frac{1}{\sqrt{2}}}\left[\begin{array}{cc}
1 & -1 \\
1 & 1
\end{array}\right]=\frac{1}{2}\left[\begin{array}{ll}
2 & 1 \\
1 & 2
\end{array}\right]
$$

The rotation matrix is obtained by Eq. (3.7.6):

$$
\mathbf{R}=\mathbf{F} \mathbf{U}^{-1}=\left[\begin{array}{cc}
-\frac{1}{2} & -1  \tag{E3.11.6}\\
1 & \frac{1}{2}
\end{array}\right]_{2}\left[\begin{array}{cc}
2 & -1 \\
-1 & 2
\end{array}\right]=\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]
$$

3.7.2 Objective Rates in Constitutive Equations. To explain why objective rates are needed for the Cauchy stress tensor, we consider the rod shown in Fig. 3.10. Suppose the simplest example of a rate constitutive equation is used, known as a hypoelastic law, where the stress rate is linearly related to the rate-ofdeformation:

$$
\begin{equation*}
\frac{D \sigma_{i j}}{D t}=C_{i j k l} D_{k l} \quad \text { or } \quad \frac{D \sigma}{D t}=\mathbf{C}: \mathbf{D} \tag{3.7.8}
\end{equation*}
$$



$$
\sigma_{x}=\sigma_{0}, \sigma_{y}=0
$$



$$
\boldsymbol{\sigma}_{x}=0, \boldsymbol{\sigma}_{y}=\sigma_{0}
$$

Fig. 3.10. Rotation of a bar under initial stress showing the change of Cauchy stress which occurs without any deformation.

The question posed here is: are the above valid constitutive equations?
The answer is negative, and can be explained as follows. Consider a solid, such as the bar in Fig. 3.10, which is stressed in its initial configuration with $\sigma_{x}=\sigma_{0}$. Now assume that the bar rotates as shown at constant length, so there is no deformation, i.e. $\mathbf{D}=0$. Recall that in rigid body motion a state of initial stress (or prestress) is frozen in the body in a solid, i.e. since the deformation does not change in a rigid body rotation, the stress as viewed by an observer riding with the body should not change. Therefore the Cauchy stress expressed in a fixed coordinate system will change during the rotation, so the material derivative of the stress must be nonzero. However, in a pure rigid body rotation, the right hand side of Eq. (3.7.8) will vanish throughout the motion, for we have already shown that the rate-of-deformation vanishes in rigid body motion. Therefore, something must be missing in Eq. (3.7.8).

The situation explained in the previous paragraph is not just hypothetical; it is representative of what happens in real situations and simulations. A body may be in a state of stress due to thermal stresses or prestressing; an example is the stress in prestressed reinforcement bars. Large rotations of an element may occur due to actual rigid body motions of the body, as in a space vehicle or a moving car, or large local large rotations, as in a buckling beam. The rotation need not be as large as 90 degrees for the same effect; we have chosen 90 degrees to simplify the numbers.

The missing factor in Eq. (3.7.8) is that it does not account for the rotation of the material. The material rotation can be accounted for correctly be using an objective rate of the stress tensor; it is also called a frame-invariant rate. We will consider three objective rates, the Jaumann rate, the Truesdell rate and the GreenNagdi rate. All of these are used in current finite element software. There are many other objective rates, some of which will be discussed in Chapter 9.
3.7.3 Jaumann rate. The Jaumann rate of the Cauchy stress is given by

$$
\begin{equation*}
\sigma^{\nabla J}=\frac{D \sigma}{D t}-\mathbf{W} \cdot \sigma-\sigma \cdot \mathbf{W}^{T} \quad \text { or } \quad \sigma_{i j}^{\nabla J}=\frac{D \sigma_{i j}}{D t}-W_{i k} \sigma_{k j}-\sigma_{i k} W_{k j}^{T} \tag{3.7.9}
\end{equation*}
$$

where $\mathbf{W}$ is the spin tensor given by Eq. (3.3.11). The superscript ${ }^{" \nabla}{ }^{\nabla}$ " here designates an objective rate; the Jaumann rate is designated by the subsequent superscript " $J$ ". One appropriate hypoelastic constitutive equation is given by

$$
\begin{equation*}
\sigma^{\nabla J}=\mathbf{C}^{J}: \mathbf{D} \quad \text { or } \quad \sigma_{i j}^{\nabla J}=C_{i j k l}^{J} D_{k l} \tag{3.7.10}
\end{equation*}
$$

The material rate for the Cauchy stress tensor, i.e. the correct equation corresponding to (3.7.8), is then

$$
\begin{equation*}
\frac{D \sigma}{D t}=\sigma^{\nabla J}+\mathbf{W} \cdot \sigma+\sigma \cdot \mathbf{W}^{T}=\mathbf{C}^{J}: \mathbf{D}+\mathbf{W} \cdot \sigma+\sigma \cdot \mathbf{W}^{T} \tag{3.7.11}
\end{equation*}
$$

where the first equality is just a rearrangement of Eq. (3.7.9) and the second equality follows from (3.7.10). We see in the above that the objective rate is a function of material response. The material derivative of the Cauchy stress then depends on two parts: the rate of change due to material response, which is reflected in the objective rate, and the change of stress due to rotation, which corresponds to the last two terms in Eq. (3.7.11).

Truesdell Rate. The Truesdell rate and Green-Naghdi rates are given in Box 3.5. The Green-Naghdi rate differs from the Jaunmann rate only in using a different measure of the rotation of the material. In the Green-Nagdi rate, the angular velocity defined in Eq. (3.2.23b) is used.


The relationship between the Truesdell rate and the Jaumann rate can be examined by replacing the velocity gradient in Eq. (3.7.23) by its symmetric and antisymmetric parts, i.e. Eq. (3.3.9):

$$
\begin{equation*}
\sigma^{\nabla \tau}=\frac{D \sigma}{D t}+\operatorname{div}(\mathbf{v}) \sigma-(\mathbf{D}+\mathbf{W}) \cdot \sigma-\sigma \cdot(\mathbf{D}+\mathbf{W})^{T} \tag{3.7.12}
\end{equation*}
$$

A comparison of Eqs. (3.7.9) and (3.7.12) then shows that the Truesdell rate includes the same spin-related terms as the Jaumann rate, but also includes additional terms which depend on the rate of deformation. To examine the relationship further, we consider a rigid body rotation for the Truesdell rate and find that

$$
\begin{equation*}
\text { when } \mathbf{D}=\mathbf{0}, \quad \sigma^{\nabla \mathcal{T}}=\frac{D \sigma}{D t}-\mathbf{W} \cdot \sigma-\sigma \cdot \mathbf{W}^{T} \tag{3.7.13}
\end{equation*}
$$

Comparison of the above with Eq. (3.7.9) shows that the Truesdell rate is equivalent to the Jaumann rate in the absence of deformation, i.e. in a rigid body rotation. However, when the Jaumann rate is used in a constitutive equation, it will give a different material rate of stress unless the constitutive equation is changed appropriately. Thus if we write the constitutive equation in the form

$$
\begin{equation*}
\sigma^{\nabla \mathcal{T}}=\mathbf{C}^{\mathcal{T}}: \mathbf{D} \tag{3.7.14}
\end{equation*}
$$

then the material response tensor $\mathbf{C}^{\mathcal{T}}$ will differ from the material response tensor associated with the Jaumann rate form of the material law in Eq. (3.7.11). For this reason, whenever the material response matrix can refer to different rates, we will often add the superscripts to specify which objective rate is used by the material law. The hypoelastic relations (3.7.11) and (3.7.14) represent the same material response if the material response tensors $\mathbf{C}^{\mathcal{T}}$ and $\mathbf{C}^{J}$ are related as follows:

$$
\begin{equation*}
\sigma^{\nabla \mathcal{T}}=\mathbf{C}^{\mathfrak{y}}: \mathbf{D}=\left(\mathbf{C}^{\mathcal{T}}+\mathbf{C}^{\sigma}\right): \mathbf{D} \tag{3.7.15}
\end{equation*}
$$

where from (3.7.12)

$$
\begin{equation*}
\mathbf{C}^{\sigma}: \mathbf{D}=(\operatorname{div} \mathbf{v}) \sigma-\mathbf{D} \cdot \sigma-\sigma \cdot \mathbf{D}^{T}=(\operatorname{tr} \mathbf{D}) \sigma-\mathbf{D} \cdot \sigma-\sigma \cdot \mathbf{D}^{T} \tag{3.7.16}
\end{equation*}
$$

The components of $\mathbf{C}^{\sigma}$ are given by

$$
\begin{equation*}
C_{i j k l}^{\sigma}=\sigma_{i j} \delta_{k l}-\delta_{i k} \sigma_{j l}-\sigma_{i l} \delta_{j k} \tag{3.7.17}
\end{equation*}
$$

With these relations, the hypoelastic relations can be modified for a Truesdell rate to match the behavior of a constitutive eqaution expressed in terms of the Jaumann rate. The correspondence to the Green-Naghdi rate depends on the difference between the angular velocity and the spin and is more difficult to adjust for.

Example 3.12 Consider a body rotating in the $x-y$ plane about the origin with an angular velocity $\omega$; the original configuration is prestressed as shown in Fig. 3.11. The motion is rigid body rotation and the related tensors are given in Example 3.2. Evaluate the material time derivative of the Cauchy stress using the Jaumann rate and integrate it to obtain the Cauchy stress as a function of time.


Figure 3.11. Rotation of a prestressed element with no deformation.
From Example 3.2, Eq. (E3.2.8) we note that

$$
\mathbf{F}=\mathbf{R}=\left[\begin{array}{cc}
c & -s  \tag{E3.12.1a}\\
s & c
\end{array}\right], \quad \dot{\mathbf{F}}=\omega\left[\begin{array}{cc}
-s & -c \\
c & -s
\end{array}\right], \quad \mathbf{F}^{-1}=\left[\begin{array}{cc}
c & s \\
-s & c
\end{array}\right]
$$

where $s=\sin \omega t, c=\cos \omega t$. The spin is evaluated in terms of the velocity gradient $\mathbf{L}$, which is given for this case by Eq. (3.3.18) and then using (E3.12.1a) :

$$
\begin{align*}
\mathbf{L}=\dot{\mathbf{F}} \cdot \mathbf{F}^{-1} & =\omega\left[\begin{array}{cc}
-s & -c \\
c & -s
\end{array}\right]\left[\begin{array}{cc}
c & s \\
-s & c
\end{array}\right]=\omega\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right] \Rightarrow \\
\mathbf{W} & =\frac{1}{2}\left(\mathbf{L}-\mathbf{L}^{T}\right)=\omega\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right] \tag{E3.12.1b}
\end{align*}
$$

The material time derivative based on the Jaumann rate is then given by | specializing (3.7.9) to the case when there is no deformation:

$$
\begin{equation*}
\frac{D \sigma}{D t}=\mathbf{W} \cdot \sigma+\sigma \cdot \mathbf{W}^{T} \tag{E3.12.1.c}
\end{equation*}
$$

( $\mathbf{D}=0$, since there is no deformation; this is easily verified by noting that the symmetric part of $\mathbf{L}$ vanishes). We now change the material time derivative to an ordinary derivative since the stress is constant in space and write out the matrices in (E3.12.1c):

$$
\begin{align*}
& \frac{d \sigma}{d t}=\omega\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]\left[\begin{array}{cc}
\sigma_{x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y}
\end{array}\right]+\left[\begin{array}{cc}
\sigma_{x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y}
\end{array}\right] \omega\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]  \tag{E3.12.2}\\
& \frac{d \sigma}{d t}=\omega\left[\begin{array}{cc}
-2 \sigma_{x y} & \sigma_{x}-\sigma_{y} \\
\sigma_{x}-\sigma_{y} & 2 \sigma_{x y}
\end{array}\right] \tag{E3.12.3}
\end{align*}
$$

It can be seen that the the material time derivative of the Cauchy stress is |symmetric. We now write out the three scalar ordinary differential equations in |three unknowns, $\sigma_{x}, \sigma_{y}$, and $\sigma_{x y}$ corresponding to (E3.12.3) (the fourth scalar |equation of the above tensor equation is omitted because of symmetry):

$$
\begin{align*}
& \frac{d \sigma_{x}}{d t}=-2 \omega \sigma_{x y}  \tag{E3.12.4a}\\
& \frac{d \sigma_{y}}{d t}=2 \omega \sigma_{x y}  \tag{E3.12.4b}\\
& \frac{d \sigma_{x y}}{d t}=\omega\left(\sigma_{x}-\sigma_{y}\right) \tag{E3.12.4c}
\end{align*}
$$

The initial conditions are

$$
\begin{equation*}
\sigma_{x}(0)=\sigma_{x}^{0}, \quad \sigma_{y}(0)=0, \quad \sigma_{x y}(0)=0 \tag{E3.12.5}
\end{equation*}
$$

It can be shown that the solution to the above differential equations is

$$
\sigma=\sigma_{x}^{0}\left[\begin{array}{ll}
c^{2} & c s  \tag{E3.12.6}\\
c s & s^{2}
\end{array}\right]
$$

We only verify the solution for $\sigma_{x}(t)$ :

$$
\begin{equation*}
\frac{d \sigma_{x}}{d t}=\sigma_{x}^{0} \frac{d\left(\cos ^{2} \omega t\right)}{d t}=\sigma_{x}^{0} \omega(-2 \cos \omega t \sin \omega t)=-2 \omega \sigma_{x y} \tag{E3.12.7}
\end{equation*}
$$

where the last step follows from the solution for $\sigma_{x y}(t)$ as given in Eq. (E3.12.7); |comparing with (E3.14.4a) we see that the differential equation is satisfied.

Examining Eq. (E3.12.6) we can see that the solution corresponds to a constant state of the corotational stress $\hat{\sigma}$, i.e. if we let the corotational stress be |given by

$$
\hat{\boldsymbol{\sigma}}=\left[\begin{array}{cc}
\sigma_{x}^{0} & 0 \\
0 & 0
\end{array}\right]
$$

then the Cauchy stress components in the global coordinate system are given by (e3.12.6) by $\sigma=\mathbf{R} \cdot \hat{\sigma} \cdot \mathbf{R}^{T}$ according to Box 3.2 with (E3.12.1a) gives the result (E3.12.6).

We leave as an exercise to show that when all of the initial stresses are nonzero, |then the solution to Eqs. (E3.12.4) is

$$
\sigma=\left[\begin{array}{cc}
c & -s  \tag{E3.12.8}\\
s & c
\end{array}\right]\left[\begin{array}{cc}
\sigma_{x}^{0} & \sigma_{x y}^{0} \\
\sigma_{x y}^{0} & \sigma_{y}^{0}
\end{array}\right]\left[\begin{array}{cc}
c & s \\
-s & c
\end{array}\right]
$$

Thus in rigid body rotation, the Jaumann rate changes the Cauchy stress so that |the corotational stress is constant. Therefore, the Jaumann rate is often called the |corotational rate of the Cauchy stress. Since the Truesdell and Green-Naghdi |rates are identical to the Jaumann rate in rigid body rotation, they also correspond to the corotational Cauchy stress in rigid body rotation.

Example 3.13 Consider an element in shear as shown in Fig. 3.12. Find the |shear stress using the Jaumann, Truesdell and Green-Naghdi rates for a |hypoelastic, isotropic material.


Figure 3.12.
The motion of the element is given by

$$
\begin{align*}
& x=X+t Y \\
& y=Y \tag{E3.13.1}
\end{align*}
$$

The deformation gradient is given by Eq. (3.2.16), so

$$
\mathbf{F}=\left[\begin{array}{ll}
1 & t  \tag{E3.13.2}\\
0 & 1
\end{array}\right], \quad \dot{\mathbf{F}}=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right], \quad \mathbf{F}^{-1}=\left[\begin{array}{cc}
1 & -t \\
0 & 1
\end{array}\right]
$$

The velocity gradient is given by Eq. (E3.12.1), and the rate-of-deformation and spin are its symmetric and skew symmetric parts so

$$
\mathbf{L}=\dot{\mathbf{F}} \mathbf{F}^{-1}=\left[\begin{array}{ll}
0 & 1  \tag{E3.13.3}\\
0 & 0
\end{array}\right] \quad \mathbf{D}=\frac{1}{2}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], \quad \mathbf{W}=\frac{1}{2}\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

The hypoelastic, isotropic constitutive equation in terms of the Jaumann rate is given by

$$
\begin{equation*}
\dot{\sigma}=\left(\lambda^{\jmath} \operatorname{trace} \mathbf{D}\right) \mathbf{I}+2 \mu^{\jmath} \mathbf{D}+\mathbf{W} \cdot \sigma+\sigma \cdot \mathbf{W}^{T} \tag{E3.13.4}
\end{equation*}
$$

We have placed the superscripts on the material constants to distinguish the material constants which are used with different objective rates. Writing out the matrices in the above gives

$$
\begin{align*}
& {\left[\begin{array}{cc}
\dot{\sigma}_{x} & \dot{\sigma}_{x y} \\
\dot{\sigma}_{x y} & \dot{\sigma}_{y}
\end{array}\right]=\mu^{g}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]} \\
& +\frac{1}{2}\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]\left[\begin{array}{cc}
\sigma_{x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{cc}
\sigma_{x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y}
\end{array}\right]\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]  \tag{E3.13.5}\\
& \dot{\sigma}_{x}=\sigma_{x y}, \quad \dot{\sigma}_{y}=-\sigma_{x y}, \quad \dot{\sigma}_{x y}=\mu^{g}+\frac{1}{2}\left(\sigma_{y}-\sigma_{x}\right) \tag{E3.13.6}
\end{align*}
$$

so

The solution to the above differential equations is

$$
\begin{equation*}
\sigma_{x}=-\sigma_{y}=\mu^{y}(1-\cos t), \quad \sigma_{x y}=\mu^{y} \sin t \tag{E3.13.7}
\end{equation*}
$$

For the Truesdell rate, the constitutive equation is

$$
\begin{equation*}
\dot{\sigma}=\lambda^{\mathcal{T}} \operatorname{tr} \mathbf{D}+2 \mu^{\tau} \mathbf{D}+\mathbf{L} \cdot \sigma+\sigma \cdot \mathbf{L}^{T}-(\operatorname{tr} \mathbf{D}) \sigma \tag{E3.13.8}
\end{equation*}
$$

This gives

$$
\begin{align*}
& {\left[\begin{array}{cc}
\dot{\sigma}_{x} & \dot{\sigma}_{x y} \\
\dot{\sigma}_{x y} & \dot{\sigma}_{y}
\end{array}\right]=\mu^{\mathrm{T}}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]} \\
& +\left[\begin{array}{cc}
0 & 1 \\
0 & 0
\end{array}\right]\left[\begin{array}{ll}
\sigma_{x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y}
\end{array}\right]+\left[\begin{array}{ll}
\sigma_{x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y}
\end{array}\right]\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] \tag{E3.13.9}
\end{align*}
$$

where we have used the results trace $\mathbf{D}=0$, see Eq. (E3.13.3). The differential equations for the stresses are

$$
\begin{equation*}
\dot{\sigma}_{x}=2 \sigma_{x y}, \quad \dot{\sigma}_{y}=0, \quad \dot{\sigma}_{x y}=\mu^{\mathrm{T}}+\sigma_{y} \tag{E3.13.10}
\end{equation*}
$$

and the solution is

$$
\begin{equation*}
\sigma_{x}=\mu^{\mathcal{T}} t^{2}, \quad \sigma_{y}=0, \quad \sigma_{x y}=\mu^{\mathcal{T}} t \tag{E3.13.11}
\end{equation*}
$$

To obtain the solution for the Cauchy stress by means of the Green-Nagdhi rate, we need to find the rotation matrix $\mathbf{R}$ by the polar decomposition theorem. To obtain the rotation, we diagonalize $\mathbf{F}^{T} \mathbf{F}$

$$
\mathbf{F}^{T} \mathbf{F}=\left[\begin{array}{cc}
1 & t  \tag{E3.13.12}\\
t & 1+t^{2}
\end{array}\right] \text {, eigenvalues } \bar{\lambda}_{i}=\frac{2+t^{2} \pm t \sqrt{4+t^{2}}}{2}
$$

The closed form solution by hand is quite involved and we recommend a computer solution. A closed form solution has been given by Dienes (1979):

$$
\begin{align*}
& \sigma_{x}=-\sigma_{y}=4 \mu^{\mathcal{G}}\left(\cos 2 \beta \operatorname{cn} \cos \beta+\beta \sin 2 \beta-\sin ^{2} \beta\right),  \tag{E3.13.13}\\
& \sigma_{x y}=2 \mu^{\mathcal{G}} \cos 2 \beta(2 \beta-2 \tan 2 \beta \operatorname{cn} \cos \beta-\tan \beta), \quad \tan \beta=\frac{t}{2} \tag{E.13.14}
\end{align*}
$$

The results are shown in Fig. 3.13.

Figure 3.13. Comparison of Objective Stress Rates
Explanation of Objective Rates. One underlying characteristic of objective rates can be gleaned from the previous example: an objective rate of the Cauchy stress instantaneously coincides with the rate of a stress field whose material rate already accounts for rotation correctly. Therefore, if we take a stress measure which rotates with the material, such as the corotational stress or the PK2 stress, and add the additional terms in its rate, then we can obtain an objective stress rate. This is not the most general framework for developing objective rates. A general framework is provided by using objectivity in the sense that the stress rate should be invariant for observers who are rotating with respect to each other. A derivation based on these principles may be found in Malvern (1969) and Truesdell and Noll (????).

To illustrate the first approach, we develop an objective rate from the corotational Cauchy stress $\hat{\sigma}$. Its material rate is given by

$$
\begin{equation*}
\frac{D \hat{\sigma}}{D t}=\frac{D\left(\mathbf{R}^{T} \sigma \mathbf{R}\right)}{D t}=\frac{D \mathbf{R}^{T}}{D t} \sigma \mathbf{R}+\mathbf{R}^{T} \frac{D \sigma}{D t} \mathbf{R}+\mathbf{R}^{T} \sigma \frac{D \mathbf{R}}{D t} \tag{3.7.18}
\end{equation*}
$$

where the first equality follows from the stress transformation in Box 3.2 and the second equality is based on the derivative of a product. If we now consider the corotational coordinate system coincident with the reference coordinates but rotating with a spin $\mathbf{W}$ then

$$
\begin{equation*}
\mathbf{R}=\mathbf{I} \quad \frac{D \mathbf{R}}{D t}=\mathbf{W}=\Omega \tag{3.7.19}
\end{equation*}
$$

Inserting the above into Eq. (3.7.18), it follows that at the instant that the corotational coordinate system coincides with the global system, the rate of the Cauchy stress in rigid body rotation is given by

$$
\begin{equation*}
\frac{D \hat{\sigma}}{D t}=\mathbf{W}^{T} \cdot \sigma+\frac{D \sigma}{D t}+\sigma \cdot \mathbf{W} \tag{3.7.20}
\end{equation*}
$$

The RHS of this expression can be seen to be identical to the correction terms in the expression for the Jaumann rate. For this reason, the Jaumann rate is often called the corotational rate of the Cauchy stress.

The Truesdell rate is derived similarly by considering the time derivative of the PK2 stress when the reference coordinates instantaneously coincide with the spatial coordinates. However, to simplify the derivation, we reverse the expressions and extract the rate corresponding to the Truesdell rate.

Readers familiar with fluid mechanics may wonder why frame-invariant rates are rarely discussed in introductory courses in fluids, since the Cauchy stress is widely used in fluid mechanics. The reason for this lies in the structure of constitutive equations which are used in fluid mechanics and in introductory fluid courses. For a Newtonian fluid, for example, $\sigma=2 \mu \mathbf{D}^{\prime}-p \mathbf{I}$, where $\mu$ is the viscosity and $\mathbf{D}^{\prime}$ is the deviatoric part of the rate-of-deformation tensor. A major difference between this constitutive equation for a Newtonian fluid and the hypoelastic law (3.7.14) can be seen immediately: the hypoelastic law gives the stress rate, whereas in the Newtonian consititutive equation gives the stress. The stress transforms in a rigid body rotation exactly like the tensors on the RHS of the equation, so this constitutive equation behaves properly in a rigid body rotation. In other words, the Newtonian fluid is objective or frame-invariant.

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Exercise ??. Consider the same rigid body rotation as in Example ??>. Find the Truesdell stress and the Green-Naghdi stress rates and compare to the Jaumann stress rate.

Starting from Eqs. (3.3.4) and (3.3.12), show that
$2 d \mathbf{x} \cdot \mathbf{D} \cdot d \mathbf{x}=2 d \mathbf{x} F^{-T} \dot{E} \dot{F}^{-1} d \mathbf{x}$
and hence that Eq. (3.3.22) holds.
Using the transformation law for a second order tensor, show that $\mathbf{R}=\hat{\mathbf{R}}$.
Using the statement of the conservation of momentum in the Lagrangian description in the initial configuration, show that it implies

$$
\mathbf{P} \mathbf{F}^{T}=\mathbf{F} \mathbf{P}^{T}
$$

Extend Example 3.3 by finding the conditions at which the Jacobian becomes negative at the Gauss quadrature points for $2 \times 2$ quadrature when the initial element is rectangular with dimension $\mathrm{a} \times \mathrm{b}$. Repeat for one-point quadrature, with the quadrature point at the center of the element.

Kinematic Jump Condition. The kinematic jump conditions are derived from the restriction that displacement remains continuous across a moving singular surface. The surface is called singular because ???. Consider a singular surface in one dimension.


Figure 3.?
Its material description is given by

$$
X=X_{S}(t)
$$

We consider a narrow band about the singular surface defined by

